

THE KOHN-LAPLACE EQUATION ON ABSTRACT CR MANIFOLDS: GLOBAL REGULARITY

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ABSTRACT. Let M be a compact, pseudoconvex-oriented, $(2n+1)$ -dimensional, abstract CR manifold of hypersurface type, $n \geq 2$. We prove the following:

- (i) If M admits a strictly CR-plurisubharmonic function on $(0, q_0)$ -forms, then the complex Green operator G_q exists and is continuous on $L^2_{0,q}(M)$ for degrees $q_0 \leq q \leq n - q_0$. In the case that $q_0 = 1$, we also establish continuity for G_0 and G_n . Additionally, the $\bar{\partial}_b$ -equation on M can be solved in $C^\infty(M)$.
- (ii) If M satisfies “a weak compactness property” on $(0, q_0)$ -forms, then G_q is a continuous operator on $H^s_{0,q}(M)$ and is therefore globally regular on M for degrees $q_0 \leq q \leq n - q_0$; and also for the top degrees $q = 0$ and $q = n$ in the case $q_0 = 1$.

We also introduce the notion of a “plurisubharmonic CR manifold” and show that it generalizes the notion of “plurisubharmonic defining function” for a domain in \mathbb{C}^N and implies that M satisfies the weak compactness property.

1. INTRODUCTION

Let M be an abstract compact smooth CR manifold of real dimension $2n+1$ equipped with a Cauchy-Riemann structure $T^{1,0}M$. The tangential Cauchy-Riemann operator $\bar{\partial}_b$ is well-defined on smooth (p, q) -forms and our interest is to understand the regularity of the canonical solution u to the $\bar{\partial}_b$ -equation, $\bar{\partial}_b u = \varphi$, i.e., the solution of the $\bar{\partial}_b$ -equation that is orthogonal to the kernel of $\bar{\partial}_b$. Our approach is via L^2 -methods and we obtain the canonical solution to the $\bar{\partial}_b$ -equation by solving the related \square_b -equation $\square_b u = f$ where the Kohn Laplacian $\square_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ and $\bar{\partial}_b^*$ is the L^2 -adjoint of $\bar{\partial}_b$ in an appropriate L^2 -space. We provide the technical details below. The operator \square_b maps $(0, q)$ -forms to $(0, q)$ -forms and its inverse on $(0, q)$ -forms (when it exists) is called the *complex Green operator* and denote G_q . Our primary goal in this paper to find sufficient conditions for the global regularity and exact regularity of G_q and related operators. Global regularity means that G_q maps smooth forms to smooth forms, and exact regularity means that G_q is continuous on all of the L^2 -Sobolev spaces $H^s_{0,q}(M)$, $s \geq 0$. Before we can investigate the exact and global regularity questions, we must generalize the existing L^2 -theory for

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solving the \square_b and $\bar{\partial}_b$ -equations in $L^2_{0,q}(M)$. Additionally, we define a family of weighted Kohn Laplacians and show that for a given index $s \geq 0$, there are weighted complex Green operators whose inverses are continuous on $H^s_{0,q}(M)$.

1.1. History – the L^2 -theory of $\bar{\partial}_b$ on CR manifolds. A classical result of Hörmander is that solvability of an operator (e.g., $\bar{\partial}_b$, \square_b , etc.) is equivalent to it having closed range. Solvability is closely linked to geometric and potential theoretic conditions on the manifold M . Curvature, for example, is measured by the Levi form. The Levi form on boundaries of domains in \mathbb{C}^{n+1} is essentially the complex analysis analog of the second fundamental form, and when it is nonnegative, M is called pseudoconvex. The first results establishing closed range of $\bar{\partial}_b$ in $L^2_{0,q}(M)$ occur when $M \subset \mathbb{C}^{n+1}$, $n \geq 1$, is pseudoconvex. Shaw [Sha85] proves that $\bar{\partial}_b$ has closed range for $0 \leq q \leq n-1$ when M is the boundary of a pseudoconvex domain. With Boas, Shaw supplements her earlier result and establishes the result in the top degree $q = n$ [BS86]. Using a microlocal argument, and in fact, developing the microlocal machinery, Kohn proves that if M is compact, pseudoconvex, and the boundary of a smooth complex manifold which admits a strictly plurisubharmonic function defined in a neighborhood of M , then $\bar{\partial}_b$ has closed range [Koh86]. In [Nic06] Nicoara extends the Kohn microlocal machine and proves that on an embedded, compact, pseudoconvex-oriented CR manifold of dimension $(2n+1) \geq 5$, $\bar{\partial}_b$ has closed range and can be solved in C^∞ .

When pseudoconvexity is relaxed, Harrington and Raich prove closed range and related estimates at the level of $(0, q)$ -forms for a fixed q , $1 \leq q \leq n-1$, by developing a weak $Y(q)$ condition that is much weaker than pseudoconvexity but still suffices [HR11, HR15]. They prove similar results to those of Nicoara [Nic06]. Harrington and Raich work on embedded manifolds in [HR11] and in a Stein manifold in [HR15]. In [HR15], they adopt Shaw's techniques to work on manifolds of minimal smoothness. In all of the above cases, a crucial property of the manifolds is the existence a strictly CR plurisubharmonic function in a neighborhood of M . Indeed, our hypotheses in Theorem 1.6 below include the existence of such a function, and our main task in proving Theorem 1.6 is to show that the arguments from the previous works hold in the generality in which we work. The main focus of this paper, however, is the global and exact regularity statements.

1.2. History – global regularity for G_q . An operator is *globally regular* if it preserves C^∞ . Determining necessary and sufficient conditions for the global regularity of the complex Green operator (and $\bar{\partial}$ -Neumann operator) is a one of the major questions in the L^2 -theory of the tangential Cauchy-Riemann operators. On compact manifolds, it is clear that if G_q preserve C^∞ locally, then it will preserve C^∞ globally. However, local regularity of G_q requires strong hypotheses, e.g., subelliptic estimates [Koh85], subelliptic estimates with multipliers [Koh00, BPZ15], or superlogarithmic estimates [Koh02, KZ11]. In parallel with this work [Kha], the first author introduces a new potential theoretical condition

named the σ -superlogarithmic property to prove the local regularity of G_q . However, this condition is far stronger than necessary to imply global regularity.

The techniques to prove global regularity do so by showing that G_q is *exactly regular*, meaning that G_q is a bounded operator from $H_{0,q}^s(M)$ to itself. There are several known conditions that suffice to prove exact (and hence global) regularity. The first is that M is the boundary of a domain that admits a plurisubharmonic defining function [BS91]. The second is when G_q is a compact operator. G_q is known to be a compact operator by Raich [Rai10] when M is a smooth, orientable, pseudoconvex CR manifold of hypersurface type of real dimension at least five that satisfies a pair of potential-theoretical conditions that he names properties $(CR-P_q)$ and $(CR-P_{n-1-q})$. Straube [Str12] improves Raich's result by shedding the orientability hypothesis and showing that $(CR-P_q)$ is equivalent to (P_q) (see [Str08] for details and background on (P_q)). Pinton, Zampieri, and Khanh further reduce the hypotheses by removing the embeddability requirement so that M is an abstract CR manifold [KPZ12]. They also prove a (real) dimension 3 compactness result under the additional assumption that $\bar{\partial}_b$ has closed range on functions. The third known condition for global and exact regularity of G_q is the existence of special 1-form that is exact on the null-space of the Levi form. Straube and Zeytuncu prove global regularity for G_q on a smooth, oriented, compact, pseudoconvex CR submanifold in \mathbb{C}^n under this hypothesis [SZ15].

The genesis of the three known sufficient conditions is that they, or their analogs, are sufficient conditions for the global/exact regularity of the $\bar{\partial}$ -Neumann operator on domains in \mathbb{C}^n . See Boas and Straube [BS99] for a discussion of the early history of the global regularity question and Straube [Str08] or Harrington [Har11] for recent work.

1.3. Main Goal and Major Results. The main goal of this paper is to prove a global regularity result which encompasses and improves on all of the existing work. In particular, our condition is weaker than Straube and Zeytuncu's in two respects: 1) we do not require exactness of the special 1-form α , only a bound in the spirit of [Str08], and 2) we no longer require M to be embedded. We state the main result below but defer the technical definitions to Section 2, though we do need the form γ that is dual to the “bad” direction T , that is, the dual to the vector that is orthogonal to the CR structure of the manifold. Also $d\gamma$ is connected to the Levi form and \mathcal{L}_λ is the analog of the complex Hessian for the function λ .

Theorem 1.1. *Let M be a smooth, compact, pseudoconvex-oriented CR manifold of dimension $2n + 1$ which admits a CR plurisubharmonic function on $(0, q_0)$ -forms, $n \geq 2$. Assume that for every $\epsilon > 0$ there exist a C^∞ real valued function λ_ϵ , a purely imaginary vector field T_ϵ , and a constant $A_\epsilon > 0$ so that $|\lambda_\epsilon|$, $\gamma(T_\epsilon)$ are uniformly bounded, $\gamma(T_\epsilon)$ is bounded away from zero, and*

$$\langle (\mathcal{L}_{\lambda_\epsilon} + A_\epsilon d\gamma) \lrcorner u, \bar{u} \rangle \geq \frac{1}{\epsilon} |\alpha_\epsilon|^2 |u|^2 \quad (1.1)$$

for any $(0, q_0)$ -forms u . The form α_ϵ is real and defined by $\alpha_\epsilon = -\{\text{Lie}\}_{T_\epsilon}(\gamma)$.

If $q_0 \leq q \leq n - q_0$, then the operators G_q , $\bar{\partial}_b G_q$, $G_q \bar{\partial}_b$, $\bar{\partial}_b^* G_q$, $G_q \bar{\partial}_b^*$, $I - \bar{\partial}_b^* \bar{\partial}_b G_q$, $I - \bar{\partial}_b^* G_q \bar{\partial}_b$, $I - \bar{\partial}_b G_q \bar{\partial}_b^*$, $\bar{\partial}_b^* G_q^2 \bar{\partial}_b$ and $\bar{\partial}_b G_q^2 \bar{\partial}_b^*$ are both globally regular and exactly regular in the L^2 -Sobolev space H^s , $s \geq 0$. In the case $q = 1$, the operators $G_0 = \bar{\partial}_b^* G_1^2 \bar{\partial}_b$ and $G_n = \bar{\partial}_b G_{n-1}^2 \bar{\partial}_b^*$ and hence G_0 , $\bar{\partial}_b G_0$, G_n , $\bar{\partial}_b^* G_n$ are both globally regular and exactly regular in H^s as well, $s \geq 0$.

Remark 1.2. The pseudoconvex-oriented is a necessary condition for working on global abstract CR manifolds containing open Levi flat sets in the sense that on such a Levi flat set we can choose local contact forms whose Levi forms are locally pseudoconvex at every point at which they are defined, but there does not exist a global contact form whose Levi form is globally pseudoconvex.

The hypothesis of Theorem 1.1 is weaker than property $(CR-P_q)$ (introduced by Raich [Rai10]). Furthermore, if $\alpha = -\{\text{Lie}\}_T(\gamma)$ is exact on the null-space of the Levi form then Straube and Zeytuncu [SZ15, Proposition 1] prove that for each $\epsilon > 0$ there exists T_ϵ such that $\alpha_\epsilon = -\{\text{Lie}\}_{T_\epsilon}(\gamma)$ satisfies $|\alpha_\epsilon| \leq \epsilon$. So if we choose $\lambda_\epsilon := \lambda$ is a strictly CR-plurisubharmonic with associated constant $A_\epsilon = 1$ as in (2.3) below, then (1.1) is satisfied for all $\epsilon > 0$, and hence the hypotheses of Theorem 1.1 holds. The following two corollaries are proven using the arguments of [SZ15].

Corollary 1.3. *Let M be a smooth, compact, pseudoconvex-oriented CR manifold of dimension $2n + 1$ which admits a CR plurisubharmonic function on $(0, 1)$ -forms, $n \geq 2$. If the real 1-form $\alpha = -\{\text{Lie}\}_T(\gamma)$ is exact on the null space of the Levi form, then the hypothesis in Theorem 1.1 holds for $(0, 1)$ -forms.*

Corollary 1.4. *Let M be a smooth, compact, pseudoconvex-oriented CR manifold of dimension $2n + 1$ which admits a CR plurisubharmonic function on $(0, 1)$ -forms, $n \geq 2$. Let S be the set of non-strictly pseudoconvex points. Suppose that at each point of S , the (real) tangent space is contained in the null space of the Levi form at the point. If the first de Rham cohomology of S is trivial, then α is exact on the null space of the Levi form (this happens if S is simply connected). Consequently, the hypotheses of Corollary 1.3 hold.*

Proof. The proof follows exactly from [SZ15, Theorem 3]. Let \mathcal{N}_x be the null space of the Levi form at x . Analogously to Lemma [BS93, Lemma on p. 230], we have $(d\alpha|_x)(X \wedge Y) = 0$ if $X, Y \in \mathcal{N}_x \oplus \overline{\mathcal{N}_x}$. Thus, $d\alpha = 0$ on S . Since the first DeRham cohomology of S is trivial, there exists h on $C^\infty(\overline{S})$ such that $\alpha = dh$ on S . This mean, α is exact on \mathcal{N}_x . \square

Also in [SZ15], the authors show that if the defining functions of M are plurisubharmonic in some neighborhood of M then α is exact on the null space of the Levi form. Their proof strongly relies on the fact that M is embedded in \mathbb{C}^n , however, we are able to remove the embeddability assumption.

Theorem 1.5. *Let M be a smooth, compact, plurisubharmonic-oriented CR manifold of dimension $2n + 1$ that admits a CR plurisubharmonic function on $(0, 1)$ -forms. Then for any $\epsilon > 0$ there exists a vector T_ϵ whose length is bounded and bounded away from 0 with associated form $\alpha_\epsilon = -\{Lie\}_{T_\epsilon}(\gamma)$ satisfying $|\alpha_\epsilon| < \epsilon$. Consequently, the conclusion of Theorem 1.1 holds for all $0 \leq q \leq n$.*

The secondary goal of this paper is to restate in a slightly more general form the L^2 -theory for $\bar{\partial}_b$ without the embeddability assumption.

Theorem 1.6. *Let M be a smooth, compact, pseudoconvex-oriented CR manifold of dimension $2n + 1$ that admits a strictly CR-plurisubharmonic function on $(0, q_0)$ -forms. If $q_0 \leq q \leq n - q_0$ then the following hold:*

(i) *The L^2 basic estimate*

$$\|u\|_{L^2}^2 \leq c(\|\bar{\partial}_b u\|_{L^2}^2 + \|\bar{\partial}_b^* u\|_{L^2}^2) \quad (1.2)$$

holds for all $u \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}_b^) \cap \mathcal{H}_{0,q}^\perp(M)$.*

(ii) *The operators $\bar{\partial}_b : L_{0,\tilde{q}}^2(M) \rightarrow L_{0,\tilde{q}+1}^2(M)$ and $\bar{\partial}_b^* : L_{0,\tilde{q}+1}^2(M) \rightarrow L_{0,\tilde{q}}^2(M)$ have closed range when $\tilde{q} = q$ or $q - 1$. Additionally, $\square_b : L_{0,q}^2(M) \rightarrow L_{0,q}^2(M)$ has closed range;*

(iii) *The operators $G_q, \bar{\partial}_b^* G_q, G_q \bar{\partial}_b^*, \bar{\partial}_b G_q, G_q \bar{\partial}_b, I - \bar{\partial}_b^* \bar{\partial}_b G_q, I - \bar{\partial}_b^* G_q \bar{\partial}_b, I - \bar{\partial}_b \bar{\partial}_b^* G_q, I - \bar{\partial}_b G_q \bar{\partial}_b^*$ are L^2 bounded. In the case $q = 1$, the operators $G_0 = \bar{\partial}_b^* G_1^2 \bar{\partial}_b$ and $G_n = \bar{\partial}_b G_{n-1}^2 \bar{\partial}_b^*$ and hence $G_0, \bar{\partial}_b G_0, G_n, \bar{\partial}_b^* G_n$ are continuous on L^2 .*

(iv) *The $\bar{\partial}_b$ -equation $\bar{\partial}_b u = \varphi$ has a solution $u \in C_{0,\tilde{q}-1}^\infty(M)$ for $\tilde{q} = q$ or $q + 1$ if φ is a $\bar{\partial}_b$ -closed, C^∞ -smooth $(0, \tilde{q})$ -form.*

(v) *The space of harmonic forms $\mathcal{H}_{0,q}(M)$ is finite dimensional.*

Remark 1.7. There is little new work to be done to prove Theorem 1.6. Our hypotheses are exactly what Nicoara [Nic06] and Harrington and Raich [HR11] use. In their work, embeddability is only used to establish the existence of a strictly CR plurisubharmonic function. We will simply highlight aspects of the earlier proofs that we need to prove Theorem 1.6.

Our main contribution is the development of an elliptic regularization that does not require M to be embedded, whereas the previous methods strongly used the embeddedness.

The outline of the rest of the paper is as follows. The technical preliminaries are given in Section 2. The establishment of a basic estimate comprises the beginning of Section 3. The remainder of the section develops the microlocal framework that we use to prove the main theorem. We prove Theorem 1.6 in Section 4. Its proof follows the argument of [HR11] (and in [Nic06]), and like [HR11], the microlocal argument proves an auxiliary result on a carefully constructed weighted spaces, Theorem 4.1. It is through the weighted spaces that we can solve $\bar{\partial}_b$ in C^∞ . We conclude the paper in Section 5 with proofs of Theorem 1.6 and Theorem 1.5.

2. DEFINITIONS

Let M be a real smooth manifold of dimension $2n + 1$, $n \geq 1$. Let $\mathbb{C}TM$ be the complexified tangent bundle over M , and $T^{1,0}M$ be a subbundle of $\mathbb{C}TM$. We say that M is a *CR manifold of hypersurface type* equipped with CR structure $T^{1,0}M$ (or CR manifold for short) if the following conditions are satisfied:

- (i) $\dim_{\mathbb{C}} T^{1,0}M = n$,
- (ii) $T^{1,0}M \cap T^{0,1}M = \{0\}$, where $T^{0,1}M = \overline{T^{1,0}M}$,
- (iii) for any $L, L' \in \Gamma(U, T^{1,0}M)$, the Lie bracket $[L, L']$ is still in $\Gamma(U, T^{1,0}M)$, where U is any open set of M and $\Gamma(U, T^{1,0}M)$ denotes the space of smooth sections of $T^{1,0}M$ over U (this condition is nonexistent when $n = 1$).

On M , we choose a Riemann metric $\langle \cdot, \cdot \rangle$ which induces a Hermitian metric on $T^{1,0}M \oplus T^{0,1}M$ so that $\langle L, \bar{L}' \rangle = 0$ for any $L, L' \in T^{1,0}M$. Given the metric $\langle \cdot, \cdot \rangle$, we choose a local frame $\{L_1, \dots, L_n\}$ of $T^{1,0}M$ and a purely imaginary vector field T that is orthogonal (and hence transversal) to $T^{1,0}M \oplus T^{0,1}M$ so that $\{L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n, T\}$ forms an orthonormal basis of $\mathbb{C}TM$. See [Bog91] for details. Denote by $\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n, \gamma$ the dual basis of 1-forms for $L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n$, and T . We call any open set $U \subset M$ a *local patch* if it admits such vectors and forms. Our interest is in global solvability, so we need a suitable partition of unity. We call a cover $\{U_\mu\}$ of M a *good cover* if each U_α is a local patch. We also let $\{\eta_\mu\}$ be a partition of unity subordinate to $\{U_\mu\}$.

For a C^2 function ϕ on M , we call the alternating $(1, 1)$ -form $\mathcal{L}_\phi = \frac{1}{2}(\partial_b \bar{\partial}_b - \bar{\partial}_b \partial_b)\phi$ on $T^{1,0}M \times T^{0,1}M$ the *Levi form of ϕ* . The Levi form \mathcal{L}_M of M at x is the Hermitian form given by $d\gamma(L_x \wedge \bar{L}'_x) = \gamma([\bar{L}'_x, L_x])$ for $L, L' \in T^{1,0}_x$. We say that M is *pseudoconvex at $x \in M$* if the Levi form is positive semidefinite in a neighborhood of x , i.e., $d\gamma(L \wedge \bar{L}) \geq 0$ for all $L \in T^{1,0}_x$. M is *orientable* if there exists a global 1-form $\tilde{\gamma}$. We say that M is *pseudoconvex-oriented* if the 1-form $\gamma \in \mathbb{C}TM^*$ is globally defined and the Levi form is positive semidefinite for all $x \in M$. All of the manifolds that we consider in this paper will be pseudoconvex-oriented. We express $d\gamma$ in a local patch U as

$$d\gamma = - \left(\sum_{i,j=1}^n c_{ij} \omega_i \wedge \bar{\omega}_j + \sum_{j=1}^n c_{0j} \gamma \wedge \omega_j + \sum_{j=1}^n \bar{c}_{0j} \gamma \wedge \bar{\omega}_j \right), \quad (2.1)$$

where the integrability condition of CR structure and the Cartan formula forces the coefficients of $\omega_i \wedge \omega_j$ and $\bar{\omega}_j \wedge \bar{\omega}_j$ to be zero. The pseudoconvex-oriented condition is equivalent to the Levi matrix $\mathcal{L}_M := \{c_{ij}\}_{i,j=1}^n$ being positive semidefinite at every $x \in M$. We extend the $n \times n$ Levi matrix to a $(n+1) \times (n+1)$ matrix $\{c_{ij}\}_{i,j=0}^n$ with entries c_{0j} for $j = 1, \dots, n$, $c_{j0} = \bar{c}_{0j}$, and c_{00} to be chosen. We say that M is *plurisubharmonic at x* if there exists c_{00} such that the extended Levi matrix $\{c_{ij}\}_{i,j=0}^n \geq 0$ in a neighborhood of x and M is *plurisubharmonic-oriented* if it is pseudoconvex-oriented and plurisubharmonic at every $x \in M$. It is obvious that if M is embedded in a Stein manifold X and admits

a plurisubharmonic defining function r then M is plurisubharmonic-oriented. Indeed, in this case the plurisubharmonic-oriented condition is fulfilled if we choose $\gamma = \frac{i}{2}(\partial r - \bar{\partial} r)$ and c_{00} is the $T := L_0 - \bar{L}_0$ component of $[L_0, \bar{L}_0]$ where $L_0 \in T^{1,0}X$ is the dual of ∂r . Let $\alpha = \sum_{j=1}^n (c_{0j}\omega_j + \bar{c}_{0j}\bar{\omega}_j)$ and observe that $\alpha = -\{\text{Lie}\}_T(\gamma)$. In [SZ15], the $(1,0)$ -form α is called *exact on the null space of the Levi form* if there exists a smooth function h , defined in a neighborhood of K , the set of weakly pseudoconvex points of M , such that

$$dh(L_z)(z) = \alpha(L_z)(z), \quad L_z \in \mathcal{N}_z \cap T_z^{1,0}, \quad z \in K,$$

where \mathcal{N}_z is the null space of the Levi form at $z \in K$. Below, we will show that if M is plurisubharmonic-oriented then α is exact on the null space of the Levi form.

Denote by dV the element of volume on M , the induced L_2 -inner product and norm on $C_{p,q}^\infty(M)$ is defined by

$$(u, v) = \int_M \langle u, \bar{v} \rangle dV, \quad \|u\|_{L^2}^2 = (u, u).$$

The function space $L_{p,q}^2(M)$ is the Hilbert space obtained by completing $C_{p,q}^\infty(M)$ under the L^2 -norm. The Sobolev spaces $H_{p,q}^s(M)$ are obtained by completing $C_{p,q}^\infty(M)$ under the usual $H^s(M)$ norm, $\|\cdot\|_{H^s}$, applied componentwise. We now want to define $\bar{\partial}_b$ on (p, q) -forms, extending our definition from functions. Define the operator $\bar{\partial}_b : C_{p,q}^\infty(M) \rightarrow C_{p,q+1}^\infty(M)$ to be the projection of the de Rham exterior differential operator d to $C_{p,q}^\infty(M)$. We denote by $\bar{\partial}_b^* : C_{p,q+1}^\infty(M) \rightarrow C_{p,q}^\infty(M)$ the L^2 -adjoint of $\bar{\partial}_b$ and define the Kohn-Laplacian by

$$\square_b := \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b : C_{p,q}^\infty(M) \rightarrow C_{p,q}^\infty(M).$$

The space of harmonic (p, q) -forms $\mathcal{H}_{p,q}(M) := \ker \square_b$ coincides with $\ker \bar{\partial}_b \cap \ker \bar{\partial}_b^*$. Our result will be to find a sufficient condition so that the estimate

$$\|u\|_{L^2} \leq c(\|\bar{\partial}_b u\|_{L^2} + \|\bar{\partial}_b^* u\|_{L^2}) \quad \text{for } u \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}_b^*) \cap \mathcal{H}_{p,q}^\perp, \quad (2.2)$$

holds and $\mathcal{H}_{p,q}(M)$ is finite dimensional. A consequence of this estimate is that \square_b has a bounded inverse on $\mathcal{H}_{p,q}^\perp(M)$. In this case, we extend the inverse to be identically 0 on $\mathcal{H}_{p,q}(M)$.

Definition 2.1. In the case that \square_b is an invertible operator on $L_{p,q}^2(M) \cap \mathcal{H}_{p,q}^\perp(M)$, we denote the inverse by $G_{p,q}$ and call it the *complex Green operator*.

The existence of a strictly CR plurisubharmonic function means that the curvature of M does not play a factor in the existence of $G_{p,q}$, so it suffices to take $p = 0$, and we denote $G_{0,q}$ by G_q .

If M is not embedded in a Stein manifold, we can define strictly CR plurisubharmonic functions as follows.

Definition 2.2. Let M be a pseudoconvex CR manifold. A C^∞ real-valued function λ defined on M is *strictly CR plurisubharmonic on $(0, q)$ -forms* if there exists a constant $a > 0$ so that so that

$$\langle (\mathcal{L}_\lambda + d\gamma) \lrcorner u, \bar{u} \rangle \geq a|u|^2, \quad (2.3)$$

for any $u \in C_{0,q}^\infty(M)$.

For u defined on U , the contraction operator \lrcorner is defined by

$$\theta \lrcorner u = \sum_{I \in \mathcal{I}_{q-1}} \left(\sum_{j=1}^n \theta_j u_{jI} \right) \bar{\omega}_I \quad \text{if } \theta = \sum_j \theta_j \omega_j \text{ is a } (1, 0)\text{-form on } U;$$

and

$$\theta \lrcorner u = \sum_{I \in \mathcal{I}_{q-1}} \sum_{j=1}^n \left(\sum_{i=1}^n \theta_{ij} u_{iI} \right) \bar{\omega}_j \wedge \bar{\omega}_I \quad \text{if } \theta = \sum_{i,j=1}^n \theta_{ij} \omega_i \wedge \bar{\omega}_j \text{ is a } (1, 1)\text{-form on } U.$$

Thus, the Levi form $d\gamma$ and \mathcal{L}_ϕ acting on $(0, q)$ -forms u, v defined in U can be expressed as

$$\langle d\gamma \lrcorner u, \bar{v} \rangle = \sum_{I \in \mathcal{I}_{q-1}} \sum_{i,j=1}^n c_{ij} u_{iI} \bar{v}_{jI} \quad \text{and} \quad \langle \mathcal{L}_\phi \lrcorner u, \bar{v} \rangle = \sum_{I \in \mathcal{I}_{q-1}} \sum_{i,j=1}^n \phi_{ij} u_{iI} \bar{v}_{jI}.$$

Remark 2.3. Strictly CR-plurisubharmonic functions always exist if M is strictly pseudoconvex or embedded into a Stein manifold. They do not, however, always exist on abstract CR manifolds. See, for example, Grauert's example [Gra63].

2.1. Working in local coordinates. Let U be a local patch of M with its associated basis of tangential vector fields $L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n, T$ and dual basis $\omega_1, \dots, \omega_n, \bar{\omega}_1, \dots, \bar{\omega}_n, \gamma$. For the moment, we work locally on U .

The condition of pseudoconvexity of M at x is equivalent to the Levi matrix $\{c_{ij}\}_{i,j=1}^n \geq 0$ in a neighborhood of x . We recall that M is *pseudoconvex-oriented* (resp., *plurisubharmonic-oriented*) if there exist a global 1-form section γ (resp., a global 1-form section γ and a smooth function c_{00}) such that the Levi matrix $\{c_{ij}\}_{i,j=1}^n$ (resp. the extended Levi matrix $\{c_{ij}\}_{i,j=0}^n$) is positive semidefinite for every $x \in M$.

We further define c_{ij}^k to be the L_k -component of $[L_i, \bar{L}_j]$. Since d applied to a $(1, 0)$ -form can produce a $(2, 0)$ -form and a $(1, 1)$ -form [Bog91, §8.2, Lemma 1], it follows from the definition of $\bar{\partial}_b$ that

$$c_{ij}^k := \omega_k([L_i, \bar{L}_j]) \underset{\text{Cartan}}{=} -\bar{\partial}_b \omega_k(L_i \wedge \bar{L}_j). \quad (2.4)$$

Therefore,

$$\bar{\partial}_b \omega_k = - \sum_{i,j=1}^n c_{ij}^k \omega_i \wedge \bar{\omega}_j. \quad (2.5)$$

and conjugating yields

$$\partial_b \bar{\omega}_k = \sum_{i,j=1}^n \bar{c}_{ji}^k \omega_i \wedge \bar{\omega}_j. \quad (2.6)$$

Using Cartan's formula again, we conclude that $-\bar{c}_{ji}^k$ coincides with the \bar{L}_k -component of $[L_i, \bar{L}_j]$. Thus the full commutator is expressed by

$$[L_i, \bar{L}_j] = c_{ij} T + \sum_{k=1}^n c_{ij}^k L_k - \sum_{k=1}^n \bar{c}_{ji}^k \bar{L}_k. \quad (2.7)$$

For a smooth function ϕ on U , we want to describe the matrix (ϕ_{ij}) of the Hermitian form $\frac{1}{2}(\partial_b \bar{\partial}_b - \bar{\partial}_b \partial_b)\phi$. Now, $\bar{\partial}_b \phi = \sum_{k=1}^n \bar{L}_k(\phi) \bar{\omega}_k$ and therefore

$$\begin{aligned} \partial_b \bar{\partial}_b \phi &= \partial_b \left(\sum_{k=1}^n \bar{L}_k(\phi) \bar{\omega}_k \right) \\ &\stackrel{\text{by (2.6)}}{=} \sum_{i,j=1}^n \left(L_i \bar{L}_j(\phi) + \sum_{k=1}^n \bar{c}_{ji}^k \bar{L}_k(\phi) \right) \omega_i \wedge \bar{\omega}_j. \end{aligned} \quad (2.8)$$

Similarly,

$$\begin{aligned} \bar{\partial}_b \partial_b \phi &= \bar{\partial}_b \left(\sum_{k=1}^n L_k(\phi) \omega_k \right) \\ &\stackrel{\text{by (2.5)}}{=} \sum_{i,j=1}^n \left(-\bar{L}_j L_i(\phi) - \sum_{k=1}^n c_{ij}^k L_k(\phi) \right) \omega_i \wedge \bar{\omega}_j. \end{aligned} \quad (2.9)$$

Combining (2.8) with (2.9) we get

$$\begin{aligned} \phi_{ij} &= \frac{1}{2} (\partial_b \bar{\partial}_b \phi - \bar{\partial}_b \partial_b \phi) (L_i \wedge \bar{L}_j) \\ &= \frac{1}{2} \left(L_i \bar{L}_j(\phi) + \bar{L}_j L_i(\phi) + \sum_{k=1}^n (\bar{c}_{ji}^k \bar{L}_k(\phi) + c_{ij}^k L_k(\phi)) \right) \\ &= \bar{L}_j L_i(\phi) + \frac{1}{2} \left([L_i, \bar{L}_j](\phi) + \sum_{k=1}^n (\bar{c}_{ji}^k \bar{L}_k(\phi) + c_{ij}^k L_k(\phi)) \right) \\ &\stackrel{\text{by (2.7)}}{=} \bar{L}_j L_i(\phi) + \frac{1}{2} c_{ij} T(\phi) + \sum_{k=1}^n c_{ij}^k L_k(\phi). \end{aligned} \quad (2.10)$$

To express a form in local coordinates, let $\mathcal{I}_q = \{(j_1, \dots, j_q) \in \mathcal{N}^q : 1 \leq j_1 < \dots < j_q \leq n\}$, and for $J \in \mathcal{I}_q$, $I \in \mathcal{I}_{q-1}$, and $j \in \mathbb{N}$, ϵ_j^{jI} be the sign of the permutation $\{j, I\} \rightarrow J$

if $\{j\} \cup I = J$ as sets, and 0 otherwise. If $u \in C_{0,q}^\infty(M)$, then u is expressed locally as a combination

$$u = \sum_{J \in \mathcal{I}_q} u_J \bar{\omega}_J,$$

of basis forms $\bar{\omega}_J = \bar{\omega}_1 \wedge \dots \wedge \bar{\omega}_{j_q}$ where $J = (j_1, \dots, j_q)$ and C^∞ -coefficients u_J .

We can also express the operator $\bar{\partial}_b : C_{0,q}^\infty(M) \rightarrow C_{0,q+1}^\infty(M)$ and its L_2 adjoint $\bar{\partial}_b^* : C_{0,q+1}^\infty(M) \rightarrow C_{0,q}^\infty(M)$ in the local basis as follows:

$$\bar{\partial}_b u = \sum_{\substack{J \in \mathcal{I}_q \\ K \in \mathcal{I}_{q+1}}} \sum_{k=1}^n \epsilon_K^{kJ} \bar{L}_k u_J \bar{\omega}_K + \sum_{\substack{J \in \mathcal{I}_q \\ K \in \mathcal{I}_{q+1}}} b_{JK} u_J \bar{\omega}_K \quad (2.11)$$

and

$$\bar{\partial}_b^* v = - \sum_{J \in \mathcal{I}_q} \left(\sum_{j=1}^n L_j v_{jK} + \sum_{K \in \mathcal{I}_{q+1}} a_{JK} v_K \right) \bar{\omega}_J \quad (2.12)$$

where $b_{JK}, a_{JK} \in C^\infty(U)$.

3. THE BASIC ESTIMATE ON CR MANIFOLDS

In this section, we will work with the weighted L_ϕ^2 -norm defined by

$$\|u\|_{L_\phi^2}^2 = (u, u)_\phi := \|ue^{-\frac{\phi}{2}}\|_{L^2}^2 = \int_M \langle u, \bar{u} \rangle e^{-\phi} dV.$$

Let $\bar{\partial}_{b,\phi}^*$ be the L_ϕ^2 -adjoint of $\bar{\partial}_b$. It is easy to see that for forms $u \in C_{0,q+1}^\infty(M)$ supported on U_μ

$$\bar{\partial}_{b,\phi}^* u = - \sum_{J \in \mathcal{I}_q} \left(\sum_{j=1}^n \sum_{K \in \mathcal{I}_{q+1}} \delta_j^\phi u_{jK} + \sum_{K \in \mathcal{I}_{q+1}} a_{JK} u_K \right) \bar{\omega}_J \quad (3.1)$$

where $\delta_j^\phi \varphi := e^\phi L_j(e^{-\phi} \varphi)$ and $a_{JK} \in C^\infty(U)$. For such u ,

$$\partial_b(\phi) \lrcorner u = -[\bar{\partial}_b^*, \phi]u = [\bar{\partial}_b, \phi]^* u = \sum_{I \in \mathcal{I}_{q-1}} \sum_{j=1}^n L_j(\phi) u_{jI} \bar{\omega}_I,$$

and hence $\bar{\partial}_{b,\phi}^* u = \bar{\partial}_b^* u - \partial_b(\phi) \lrcorner u$. Furthermore,

$$\begin{aligned} [\delta_i^\phi, \bar{L}_j] &= \bar{L}_j L_i(\phi) + [L_i, \bar{L}_j] \\ &\stackrel{\text{by (2.10) and (2.7)}}{=} \phi_{ij} - \frac{1}{2} c_{ij} T(\phi) - \sum_{k=1}^n c_{ij}^k L_k(\phi) + c_{ij} T - \sum_{k=1}^n \bar{c}_{ji}^k \bar{L}_k + \sum_{k=1}^n c_{ij}^k L_k \\ &= \phi_{ij} + c_{ij} T - \sum_{k=1}^n \bar{c}_{ji}^k \bar{L}_k + \sum_{k=1}^n c_{ij}^k \delta_k^\phi - \frac{1}{2} c_{ij} T(\phi). \end{aligned} \quad (3.2)$$

The equalities (2.11) and (3.1) lead us to Kohn-Morrey-Hörmander inequality or basic estimate for CR manifolds. It does not function quite in the same manner as the Kohn-Morrey-Hörmander inequality on domains because it cannot be applied directly to prove closed range estimates. The terms involving T require significant effort to estimate. In fact, estimating the T terms are the heart of the proof of Theorem 1.6. Equations similar in spirit to Theorem 3.1 have appeared before (e.g., [HR11, Equation (12) and (10)]) but the earlier versions do not apply to as wide of a class of CR manifolds as we consider here. We will not need it here, but we could write Theorem 3.1 even more generally by using the weak $Y(q)$ technology (namely, the form Υ) from [HR15]. We do not do that here for expositional clarity.

Theorem 3.1. *Let M be a CR manifold and U a local patch. Let ϕ be a real C^2 function and q_0 be an integer with $0 \leq q_0 \leq n$. There exists a constant C (independent of ϕ) such that for any $u \in C_{0,q}^\infty(M)$ with support in U ,*

$$\begin{aligned} \|\bar{\partial}_b u\|_{L_\phi^2}^2 + \|\bar{\partial}_{b,\phi}^* u\|_{L_\phi^2}^2 + C\|u\|_{L_\phi^2}^2 &\geq \frac{1}{2} \left(\sum_{j=1}^{q_0} \|\delta_j^\phi u\|_{L_\phi^2}^2 + \sum_{j=q_0+1}^n \|\bar{L}_j u\|_{L_\phi^2}^2 \right) \\ &+ \sum_{I \in \mathcal{I}_{q-1}} \sum_{i,j=1}^n (\phi_{ij} u_{iI}, u_{jI})_\phi - \sum_{J \in \mathcal{I}_q} \sum_{j=1}^{q_0} (\phi_{jj} u_J, u_J)_\phi \\ &+ \operatorname{Re} \left\{ \sum_{I \in \mathcal{I}_{q-1}} \sum_{i,j=1}^n (c_{ij} T u_{iI}, u_{jI})_\phi - \sum_{J \in \mathcal{I}_q} \sum_{j=1}^{q_0} (c_{jj} T u_J, u_J)_\phi \right\}. \end{aligned}$$

Proof. We use (2.11) and compute

$$\begin{aligned} \|\bar{\partial}_b u\|_{L_\phi^2}^2 &= \sum_{\substack{K \in \mathcal{I}_{q+1} \\ J, J' \in \mathcal{I}_q}} \sum_{k,k'=1}^n \epsilon_{k'J'}^{kJ} (\bar{L}_k u_J, \bar{L}_{k'} u_{J'})_\phi + \sum_{\substack{K \in \mathcal{I}_{q+1} \\ J, J' \in \mathcal{I}_q}} (b_{JK} u_J, b_{J'K} u_{J'})_\phi \\ &+ 2 \operatorname{Re} \left[\sum_{\substack{K \in \mathcal{I}_{q+1} \\ J, J' \in \mathcal{I}_q}} \sum_{k=1}^n \epsilon_K^{kJ} (\bar{L}_k u_J, a_{J'K} u_{J'})_\phi \right]. \end{aligned}$$

If $\epsilon_{k'J}^{kJ} \neq 0$, then either $k = k'$ and $J = J'$ or $J = \{k\} \cup I$ and $J' = \{k'\} \cup I$ for some $I \in \mathcal{I}_{q-1}$. In the latter case $\epsilon_{k'J}^{kJ} = \epsilon_{k'kI}^{kk'I} = -1$. We also use the notation $(\bar{L}u, u)_\phi$ to denote any term of the form $(a\bar{L}_k u_J, u_{J'})_\phi$ or its conjugate where $a \in C^\infty(U_\mu)$. We use $(\delta^\phi u, u)_\phi$ to denote any term of the same form with δ_j^ϕ replacing \bar{L}_k . It therefore follows that

$$\begin{aligned} \|\bar{\partial}_b u\|_{L_\phi^2}^2 &= \sum_{J \in \mathcal{I}_q} \sum_{k \notin J} \|\bar{L}_k u_J\|_{L_\phi^2}^2 - \sum_{I \in \mathcal{I}_{q-1}} \sum_{\substack{k, k'=1 \\ k \neq k'}}^n (\bar{L}_k u_{k'I}, \bar{L}_{k'} u_{kI})_\phi + (\bar{L}u, u)_\phi + O(\|u\|_{L_\phi^2}^2) \\ &= \sum_{J \in \mathcal{I}_q} \sum_{k=1}^n \|\bar{L}_k u_J\|_{L_\phi^2}^2 - \sum_{I \in \mathcal{I}_{q-1}} \sum_{k, k'=1}^n (\bar{L}_k u_{k'I}, \bar{L}_{k'} u_{kI})_\phi + (\bar{L}u, u)_\phi + O(\|u\|_{L_\phi^2}^2). \end{aligned}$$

A similar (but simpler) calculation shows that

$$\|\bar{\partial}_{b,\phi}^* u\|_{L_\phi^2}^2 = \sum_{I \in \mathcal{I}_{q-1}} \sum_{j, j'=1}^n (\delta_j^\phi u_{jI}, \delta_{j'}^\phi u_{j'I})_\phi + (\delta^\phi u, u)_\phi + O(\|u\|_{L_\phi^2}^2)$$

To proceed next, we integrate by parts and observe that

$$(\bar{L}_k u_J, \bar{L}_{k'} u_{J'})_\phi = (\delta_{k'}^\phi u_J, \delta_k^\phi u_{J'})_\phi - ([\delta_{k'}^\phi, \bar{L}_k] u_J, u_{J'})_\phi + (\bar{L}u, u)_\phi + (\delta^\phi u, u)_\phi + O(\|u\|_{L_\phi^2}^2). \quad (3.3)$$

An immediate consequence of this equality is that

$$-(\bar{L}_k u_{jI}, \bar{L}_j u_{kI})_\phi + (\delta_j^\phi u_{jI}, \delta_k^\phi u_{kI})_\phi = ([\delta_j^\phi, \bar{L}_k] u_{jI}, u_{kI})_\phi + (\bar{L}u, u)_\phi + (\delta^\phi u, u)_\phi + O(\|u\|_{L_\phi^2}^2)$$

and therefore

$$\begin{aligned} \|\bar{\partial}_b u\|_{L_\phi^2}^2 + \|\bar{\partial}_{b,\phi}^* u\|_{L_\phi^2}^2 &= \sum_{J \in \mathcal{I}_q} \sum_{k=1}^n \|\bar{L}_k u_J\|_{L_\phi^2}^2 + \operatorname{Re} \left\{ \sum_{I, I' \in \mathcal{I}_{q-1}} \sum_{j, k=1}^n ([\delta_j^\phi, \bar{L}_k] u_{jI}, u_{kI})_\phi \right\} \quad (3.4) \\ &\quad + (\bar{L}u, u)_\phi + (\delta^\phi u, u)_\phi + O(\|u\|_{L_\phi^2}^2). \end{aligned}$$

Finishing the proof requires four observations. First, using (3.2) on the terms $([\delta_j^\phi, \bar{L}_k] u_{jI}, u_{kI})_\phi$ produces the off-diagonal terms involving ϕ_{jk} and $c_{jk}T$. Second, the on-diagonal terms appear when (3.3) is applied to $\|L_j u_J\|_{L_\phi^2}^2$ for $1 \leq j \leq q_0$. Third, we must control $(\bar{L}u, u)_\phi$ and $(\delta^\phi u, u)_\phi$, but this is a simple matter of recognizing that $(\bar{L}u, u)_\phi = (\delta^\phi u, u)_\phi + O(\|u\|_{L_\phi^2}^2)$ so we can absorb all of these terms using a small constant/large constant argument where we pay the price of reducing the coefficient of the “gradient” terms to 1/2 and increasing $O(\|u\|_{L_\phi^2}^2)$. We have the result, except that it is not yet clear that the $O(\|u\|_{L_\phi^2}^2)$ term is independent of ϕ because the term $c_{jk}T(\phi)$ appears in (3.2). However, $\{c_{jk}\}_{j,k=1}^n$ is a positive semidefinite matrix and hence has real eigenvalues, and T is a purely imaginary

operator. This means

$$\operatorname{Re} \left\{ \sum_{j,k=1}^n (c_{jk} T(\phi) u_{jI}, u_{kI})_{\phi} \right\} = 0.$$

The $T(\phi)$ terms that appear from the integration by parts in the second observation (the one regarding the on-diagonal terms) are handled identically. \square

The difference between the Kohn-Morrey-Hörmander estimate for domains (see, e.g., [Str10] or [CS01]) and Theorem 3.1 is the presence of T instead of a boundary integral. It is for estimating the T term that we use a microlocal argument. Specifically, the estimate for T uses a consequence of the sharp Gårding inequality. Recall the formulation from [Rai10].

Proposition 3.2. *Let R be a first order pseudodifferential operator such that $\sigma(R) \geq \kappa$ where κ is some positive constant and (h_{jk}) an $n \times n$ hermitian matrix (that does not depend on ξ). Then there exists a constant C such that if the sum of any q eigenvalues of (h_{jk}) is nonnegative, then*

$$\operatorname{Re} \left\{ \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^n (h_{jk} R u_{jI}, u_{kI}) \right\} \geq \kappa \operatorname{Re} \sum_{I \in \mathcal{I}_{q-1}} \sum_{j,k=1}^n (h_{jk} u_{jI}, u_{kI}) - C \|u\|_{L^2}^2,$$

and if the sum of any collection of $(n - q)$ eigenvalues of (h_{jk}) is nonnegative, then

$$\begin{aligned} & \operatorname{Re} \left\{ \sum_{J \in \mathcal{I}_q} \sum_{j=1}^n (h_{jj} R u_J, u_J) - \sum_{H \in \mathcal{I}_{q-1}} \sum_{j,k=1}^n (h_{jk} R u_{jH}, u_{kH}) \right\} \\ & \geq \kappa \operatorname{Re} \left\{ \sum_{J \in \mathcal{I}_q} \sum_{j=1}^n (h_{jj} u_J, u_J) - \sum_{H \in \mathcal{I}_{q-1}} \sum_{j,k=1}^n (h_{jk} u_{jH}, u_{kH}) \right\} - C \|u\|_{L^2}^2. \end{aligned}$$

Note that (h_{jk}) may be a matrix-valued function in z but may not depend on ξ .

3.1. Microlocal analysis – the setup. To bound the terms from Theorem 3.1 that involve T , we continue to work on smooth forms that are supported in a small neighborhood $U \subset M$. Our approach is microlocal and we adopt the familiar setup introduced by Kohn [Koh86, Koh02]. See also Nicoara [Nic06] and Raich [Rai10].

Denote the coordinates \mathbb{R}^{2n+1} by $x = (x', x_{2n+1}) = (x_1, \dots, x_{2n}, x_{2n+1})$ with the origin at some $x_0 \in U$. We can arrange the coordinates so that if $z_j = \frac{x_j + \sqrt{-1}x_{j+n}}{\sqrt{-1}}$ for $j = 1, \dots, n$, then $L_j|_{z_0} = \frac{\partial}{\partial z_j}|_{z_0}$ for $j = 1, \dots, n$, and $T = -\sqrt{-1} \frac{\partial}{\partial x_{2n+1}}$. Let $\xi = (\xi_1, \dots, \xi_{2n+1}) = (\xi', \xi_{2n+1})$ be the dual coordinates to x in Fourier space.

Let $\mathcal{C}^+, \mathcal{C}^-, \mathcal{C}^0$ be a covering of \mathbb{R}^{2n+1} so that

$$\begin{aligned}\mathcal{C}^+ &= \{\xi : \xi_{2n+1} > \frac{1}{4}|\xi'|\} \cap \{\xi : |\xi| \geq 1\}; \\ \mathcal{C}^- &= \{\xi : \xi \in \mathcal{C}^+\}; \\ \mathcal{C}^0 &= \{\xi : |\xi_{2n+1}| < \frac{3}{4}|\xi'|\} \cup \{\xi : |\xi| < 3\}.\end{aligned}\tag{3.5}$$

For the remainder of this paper, let ψ be a smooth function so that $\psi \equiv 1$ on $\{\xi : \xi_{2n+1} > \frac{1}{3}|\xi'|\} \cap \{\xi : |\xi| \geq 2\}$ and $\text{supp } \psi \subset \mathcal{C}^+$. It follows from the definitions of $\mathcal{C}^+, \mathcal{C}^0$, and ψ , that $\text{supp } d\psi \subset \mathcal{C}^0$. Define

$$\psi^+(\xi) := \psi(\xi), \quad \psi^-(\xi) := \psi(-\xi), \quad \psi^0(\xi) := \sqrt{1 - (\psi^+(\xi))^2 - (\psi^-(\xi))^2}.$$

Let $\tilde{\psi}^0$ be a smooth function that dominates ψ^0 in the sense that $\text{supp } \tilde{\psi}^0 \subset \mathcal{C}^0$ and $\tilde{\psi}^0 = 1$ on a neighborhood of $\text{supp } \psi^0 \cup \text{supp } (d\psi^+) \cup \text{supp } (d\psi^-)$.

Associated to the smooth function ψ is a pseudodifferential operator Ψ whose symbol $\sigma(\Psi) = \psi$. This means that if $\varphi \in C_c^\infty(U)$, then

$$\widehat{\Psi\varphi}(\xi) = \psi(\xi)\hat{\varphi}(\xi),$$

where $\hat{\cdot}$ denotes the Fourier transform. The operators Ψ^+, Ψ^-, Ψ^0 , and $\tilde{\Psi}^0$ are defined analogously, with symbols ψ^+, ψ^-, ψ^0 , and $\tilde{\psi}^0$, respectively.

By construction, $(\psi^+)^2 + (\psi^-)^2 + (\psi^0)^2 = 1$, from which it follows immediately that $(\Psi^+)^*\Psi^+ + (\Psi^-)^*\Psi^- + (\Psi^0)^*\Psi^0 = Id$, the identity operator.

For the proof of Theorem 1.6, we will need dilated versions Ψ^\bullet and $\tilde{\Psi}^0$ where the superscript \bullet means $+, -, \text{ or } 0$. Let $A \geq 1$ (chosen later). Let Ψ_A^\bullet and $\tilde{\Psi}_A^0$ be the pseudodifferential operators with symbol $\psi_A^\bullet(\xi) = \psi^\bullet(\xi/A)$ and $\tilde{\psi}_A^0(\xi) = \tilde{\psi}^0(\xi/A)$, respectively. We say that a cutoff function ζ dominates a cutoff function ζ' and denote by $\zeta' \prec \zeta$ if $\zeta \equiv 1$ on $\text{supp } \zeta'$. We write the next several results for a generic A but will use $A := t$ in the proof of Theorem 1.1 and $A := A_\epsilon$ in the proof of Theorem 1.5. The next result follows immediately from Proposition 3.2 and the arguments of Lemma 4.6 and Lemma 4.7 in [Rai10].

Lemma 3.3. *Let M be a pseudoconvex CR manifold and U be a local patch of M . For $\phi \in C^\infty(M)$, $u \in C_{0,q}^\infty(M)$, and cutoff functions $\zeta \prec \tilde{\zeta} \prec \zeta'$ on U , we have*

$$\begin{aligned}(i) \quad \text{Re} \Big((d\gamma \lrcorner T\tilde{\zeta}\Psi_A^+\zeta u, \tilde{\zeta}\Psi_A^+\zeta u)_\phi \Big) &\geq A(d\gamma \lrcorner \tilde{\zeta}\Psi_A^+\zeta u, \tilde{\zeta}\Psi_A^+\zeta u)_\phi \\ &\quad - c\|\tilde{\zeta}\Psi_A^+\zeta u\|_{L_\phi^2}^2 - c_{A,\phi}\|\zeta'\tilde{\Psi}_A^0\zeta u\|_{L^2}^2\end{aligned}\tag{3.6}$$

for any $q = 1, \dots, n$; and

$$\begin{aligned}
(ii) \quad & \operatorname{Re} \left((d\gamma \lrcorner T\tilde{\zeta}\Psi_A^- \zeta u, \tilde{\zeta}\Psi_A^- \zeta u)_{-\phi} - (\operatorname{Tr}(d\gamma)T\tilde{\zeta}\Psi_A^- \zeta u, \tilde{\zeta}\Psi_A^- \zeta u)_{-\phi} \right) \\
& \geq A \left(\operatorname{Tr}(d\gamma)\tilde{\zeta}\Psi_A^- \zeta u, \tilde{\zeta}\Psi_A^- \zeta u \right)_{-\phi} - (d\gamma \lrcorner \tilde{\zeta}\Psi_A^- \zeta u, \tilde{\zeta}\Psi_A^- \zeta u)_{-\phi} \\
& \quad - c \|\tilde{\zeta}\Psi_A^- \zeta u\|_{L^2_{-\phi}}^2 - c_{A,\phi} \|\zeta' \tilde{\Psi}_A^0 \zeta u\|_{L^2}^2
\end{aligned} \tag{3.7}$$

for any $q = 0, 1, \dots, n-1$. Here, c (resp. $c_{A,\phi}$) is a positive constant independent (resp. dependent) of ϕ .

In combination with the Kohn-Morrey-Hömander inequality, Lemma 3.3 yields

Corollary 3.4. *Let M be a pseudoconvex CR manifold and U be a local patch of M . For $\phi \in C^\infty(M)$, $u \in C_{0,q}^\infty(M)$, and cutoff function $\zeta \prec \tilde{\zeta} \prec \zeta'$ on U , then we have*

$$\begin{aligned}
c_{A,\phi} \|\zeta' \tilde{\Psi}_A^0 \zeta u\|_{L^2}^2 + c \left(\|\tilde{\zeta}\Psi_A^+ \zeta u\|_{L^2_\phi}^2 + \|\bar{\partial}_b \tilde{\zeta}\Psi_A^+ \zeta u\|_{L^2_\phi}^2 + \|\bar{\partial}_{b,\phi}^* \tilde{\zeta}\Psi_A^+ \zeta u\|_{L^2_\phi}^2 \right) \\
\geq \left((\mathcal{L}_\phi + Ad\gamma) \lrcorner \tilde{\zeta}\Psi_A^+ \zeta u, \tilde{\zeta}\Psi_A^+ \zeta u \right)_\phi
\end{aligned} \tag{3.8}$$

for any $n = 1, \dots, n$; and

$$\begin{aligned}
c_{A,\phi} \|\zeta' \tilde{\Psi}_A^0 \zeta u\|_{L^2}^2 + c \left(\|\tilde{\zeta}\Psi_A^- \zeta u\|_{L^2_{-\phi}}^2 + \|\bar{\partial}_b \tilde{\zeta}\Psi_A^- \zeta u\|_{L^2_{-\phi}}^2 + \|\bar{\partial}_{b,-\phi}^* \tilde{\zeta}\Psi_A^- \zeta u\|_{L^2_{-\phi}}^2 \right) \\
\geq \left([\operatorname{Tr}(\mathcal{L}_\phi) + A \operatorname{Tr}(d\gamma)] \times \tilde{\zeta}\Psi_A^- \zeta u, \tilde{\zeta}\Psi_A^- \zeta u \right)_{-\phi} \\
- \left([\mathcal{L}_\phi + Ad\gamma] \lrcorner \tilde{\zeta}\Psi_A^- \zeta u, \tilde{\zeta}\Psi_A^- \zeta u \right)_{-\phi}
\end{aligned} \tag{3.9}$$

for any $q = 0, \dots, n-1$.

4. THE L^2 -SOBOLEV THEORY FOR \square_b AND THE PROOF OF THEOREM 1.6

4.1. The existence to the solution of the \square_b when $n \geq 2$ and $1 \leq q \leq n-1$. We now assume that M is endowed with a smooth function λ that is strictly CR-plurisubharmonic on $(0, q_0)$ -forms whose defining inequality is given by (2.3). The function λ is q_0 -compatible in the language of Harrington and Raich [HR11], and we can follow their argument nearly verbatim to establish a weighted L^2 -theory (compare with the proof of [HR11, Theorem 1.2]).

Observe that if the inequality (2.3) holds for q_0 , then it holds for any $q \geq q_0$. Additionally,

$$\langle [\operatorname{Tr}(\mathcal{L}_\lambda) + \operatorname{Tr}(d\gamma)]v, \bar{v} \rangle - \langle (\mathcal{L}_\lambda + d\gamma) \lrcorner v, \bar{v} \rangle \geq a|v|^2 \tag{4.1}$$

for all $(0, q)$ -forms v with $q \leq n - q_0$.

For each $t \geq 1$, we use Corollary 3.4 with $\phi = t\lambda$ to obtain

$$\begin{aligned} c_t \|\zeta' \tilde{\Psi}_t^0 \zeta u\|_{L^2}^2 + c \left(\|\tilde{\zeta} \Psi_t^+ \zeta u\|_{L_{t\lambda}^2}^2 + \|\bar{\partial}_b \tilde{\zeta} \Psi_t^+ \zeta u\|_{L_{t\lambda}^2}^2 + \|\bar{\partial}_{b,t\lambda}^* \tilde{\zeta} \Psi_t^+ \zeta u\|_{L_{t\lambda}^2}^2 \right) \\ \geq \left((\mathcal{L}_{t\lambda} + td\gamma) \lrcorner \tilde{\zeta} \Psi_t^+ \zeta u, \tilde{\zeta} \Psi_t^+ \zeta u \right)_{t\lambda} \\ \geq at \|\tilde{\zeta} \Psi_t^+ \zeta u\|_{L_{t\lambda}^2}^2 \end{aligned} \quad (4.2)$$

for any $u \in C_{0,q}^\infty(M)$ with $q \geq q_0$. Analogously, we also have

$$\begin{aligned} c_t \|\zeta' \tilde{\Psi}_t^0 \zeta u\|_{L^2}^2 + c \left(\|\tilde{\zeta} \Psi_t^- \zeta u\|_{L_{-t\lambda}^2}^2 + \|\bar{\partial}_b \tilde{\zeta} \Psi_t^- \zeta u\|_{L_{-t\lambda}^2}^2 + \|\bar{\partial}_{b,-t\lambda}^* \tilde{\zeta} \Psi_t^- \zeta u\|_{L_{-t\lambda}^2}^2 \right) \\ \geq at \|\tilde{\zeta} \Psi_t^- \zeta u\|_{L_{-t\lambda}^2}^2 \end{aligned} \quad (4.3)$$

for any $u \in C_{0,q}^\infty(M)$ with $q \leq n - q_0$. This estimate holds for cutoff functions $\zeta, \tilde{\zeta}, \zeta'$ having compact support on a local patch U of M . In order to prove a global estimate, we let $\{U_\nu\}$ be a cover of M and $\{\zeta_\nu\}$ be a partition of unity subordinate to $\{U_\nu\}$. Supported on each U_ν are the pseudodifferential operators $\Psi_t^{\cdot,\nu}$ and $\tilde{\Psi}_t^{0,\nu}$ where \cdot represents $+$, $-$, or 0 . For each ζ_ν , let $\tilde{\zeta}_\nu$ be a cutoff function that dominates ζ_ν . We define an inner product and norm that are well-suited to estimates using microlocal analysis. Set

$$\langle u, v \rangle_{t\lambda} = \sum_\nu \left[(\tilde{\zeta}_\nu \Psi_t^+ \zeta_\nu u^\nu, \tilde{\zeta}_\nu \Psi_t^+ \zeta_\nu v^\nu)_{L_{t\lambda}^2} + (\tilde{\zeta}_\nu \Psi_t^0 \zeta_\nu u^\nu, \tilde{\zeta}_\nu \Psi_t^0 \zeta_\nu v^\nu) + (\tilde{\zeta}_\nu \Psi_t^- \zeta_\nu u^\nu, \tilde{\zeta}_\nu \Psi_t^- \zeta_\nu v^\nu)_{-t\lambda} \right]$$

and

$$\langle |u| \rangle_{t\lambda}^2 = \sum_\nu \left[\|\tilde{\zeta}_\nu \Psi_t^+ \zeta_\nu u^\nu\|_{L_{t\lambda}^2}^2 + \|\tilde{\zeta}_\nu \Psi_t^0 \zeta_\nu u^\nu\|_{L^2}^2 + \|\tilde{\zeta}_\nu \Psi_t^- \zeta_\nu u^\nu\|_{L_{-t\lambda}^2}^2 \right],$$

where u^ν is the form u expressed in the local coordinates on U_ν . The superscript ν will often be omitted. We denote the adjoint of $\bar{\partial}_b$ with respect to this norm by $\bar{\partial}_b^{*,t}$ and the associated quadratic form

$$Q_{b,t\lambda} \langle |u, v| \rangle = \langle |\bar{\partial} u, \bar{\partial} v| \rangle_{t\lambda} + \langle |\bar{\partial}_b^{*,t} u, \bar{\partial}_b^{*,t} v| \rangle_{t\lambda}.$$

The space of $t\lambda$ -harmonic forms $\mathcal{H}_{t\lambda}^q(M)$ is

$$\mathcal{H}_{t\lambda}^q(M) = \{u \in L_{0,q}^2(M) : Q_{b,t\lambda} \langle |u, u| \rangle = 0\}.$$

By using the pseudoconvex-oriented hypothesis, the estimates (4.2)-(4.3) for U_ν , and the well-known elliptic estimate for Ψ^0 , it follows that there exists $T_0 > 0$ such that for any $t \geq t_0$ the estimate

$$\langle |u| \rangle_{t\lambda}^2 \leq \frac{c}{t} Q_{b,t\lambda} \langle |u, u| \rangle + c_t \|u\|_{H^{-1}}^2 \quad (4.4)$$

holds for all $u \in C_{0,q}^\infty(M)$ with $q_0 \leq q \leq n - q_0$. See [Nic06, HR11] for details.

For a form u on M , the Sobolev norm of order s is given by the following:

$$\|u\|_{H^s}^2 = \sum_\nu \|\tilde{\zeta}_\nu \Lambda^s \zeta_\nu u^\nu\|_{L^2}^2$$

where Λ is defined to be the pseudodifferential operator with symbol $(1 + |\xi|^2)^{1/2}$.

As in [HR11], we can also bring the estimate (4.4) to higher order Sobolev indices: for each $s \geq 0$, there exists $T_s > 0$ such that for any $t \geq T_s$,

$$\langle |\Lambda^s u| \rangle_{t\lambda}^2 \leq \frac{c}{t} (\langle |\Lambda^s \bar{\partial}_b u| \rangle_{t\lambda}^2 + \langle |\Lambda^s \bar{\partial}_b^{*,t} u| \rangle_{t\lambda}^2) + c_t \|u\|_{H^{s-1}}^2 \quad (4.5)$$

holds for all $u \in C_{0,q}^\infty(M)$ with $q_0 \leq q \leq n - q_0$. In [Nic06], it is shown that there exist constants c_t and C_t so that

$$c_t \|u\|_{L^2}^2 \leq \langle |u| \rangle_{t\lambda}^2 \leq C_t \|u\|_{L^2}^2 \quad (4.6)$$

where c_t and C_t depend on $\max_M |\lambda|$. We thus have closed range estimates for $\bar{\partial}_b : H_{0,q}^s(M) \rightarrow H_{0,q+1}^s(M)$ and $\bar{\partial}_b^{*,t} : H_{0,q}^s(M) \rightarrow H_{0,q-1}^s(M)$. The following theorem now follows from the arguments of [HR11].

Theorem 4.1. *Let M^{2n+1} be an abstract CR manifold that is pseudoconvex-oriented and admits a smooth function λ that is strictly CR plurisubharmonic on $(0, q_0)$ -forms for some $1 \leq q_0 \leq \frac{n}{2}$. Then for all $q_0 \leq q \leq n - q_0$ and $s \geq 0$, there exists $T_s \geq 0$ so that the following hold:*

- (i) *The operators $\bar{\partial}_b : L_{0,q}^2(M) \rightarrow L_{0,q+1}^2(M)$ and $\bar{\partial}_b : L_{0,q-1}^2(M) \rightarrow L_{0,q}^2(M)$ have closed range with respect to $\langle |\cdot| \rangle_{t\lambda}$. Additionally, for any $s > 0$ if $t \geq T_s$, then $\bar{\partial}_b : H_{0,q}^s(M) \rightarrow H_{0,q+1}^s(M)$ and $\bar{\partial}_b : H_{0,q-1}^s(M) \rightarrow H_{0,q}^s(M)$ have closed range with respect to $\langle |\Lambda^s \cdot| \rangle_{t\lambda}$;*
- (ii) *The operators $\bar{\partial}_b^{*,t} : L_{0,q+1}^2(M) \rightarrow L_{0,q}^2(M)$ and $\bar{\partial}_b^{*,t} : L_{0,q}^2(M) \rightarrow L_{0,q-1}^2(M)$ have closed range with respect to $\langle |\cdot| \rangle_{t\lambda}$. Additionally, if $t \geq T_s$, then $\bar{\partial}_b^{*,t} : H_{0,q+1}^s(M) \rightarrow H_{0,q}^s(M)$ and $\bar{\partial}_b^{*,t} : H_{0,q}^s(M) \rightarrow H_{0,q-1}^s(M)$ have closed range with respect to $\langle |\Lambda^s \cdot| \rangle_{t\lambda}$;*
- (iii) *The Kohn Laplacian defined by $\square_b^t = \bar{\partial}_b \bar{\partial}_b^{*,t} + \bar{\partial}_b^{*,t} \bar{\partial}_b$ has closed range on $L_{0,q}^2(M)$ (with respect to $\langle |\cdot| \rangle_{t\lambda}$) and also on $H_{0,q}^s(M)$ (with respect to $\langle |\Lambda^s \cdot| \rangle_{t\lambda}$) if $t \geq T_s$;*
- (iv) *The space of harmonic forms $\mathcal{H}_t^q(M)$, defined to be the $(0, q)$ -forms annihilated by $\bar{\partial}_b$ and $\bar{\partial}_b^{*,t}$ is finite dimensional;*
- (v) *The complex Green operator $G_{q,t}$ is continuous on $L_{0,q}^2(M)$ (with respect to $\langle |\cdot| \rangle_{t\lambda}$) and also on $H_{0,q}^s(M)$ (with respect to $\langle |\Lambda^s \cdot| \rangle_{t\lambda}$) if $t \geq T_s$;*
- (vi) *The canonical solution operators for $\bar{\partial}_b$, $\bar{\partial}_b^{*,t} G_{q,t} : L_{0,q}^2(M) \rightarrow L_{0,q-1}^2(M)$ and $G_{q,t} \bar{\partial}_b^{*,t} : L_{0,q+1}^2(M) \rightarrow L_{0,q}^2(M)$ are continuous (with respect to $\langle |\cdot| \rangle_{t\lambda}$). Additionally, $\bar{\partial}_b^{*,t} G_{q,t} : H_{0,q}^s(M) \rightarrow H_{0,q-1}^s(M)$ and $G_{q,t} \bar{\partial}_b^{*,t} : H_{0,q+1}^s(M) \rightarrow H_{0,q}^s(M)$ are continuous (with respect to $\langle |\Lambda^s \cdot| \rangle_{t\lambda}$) if $t \geq T_s$.*
- (vii) *The canonical solution operators for $\bar{\partial}_b^{*,t}$, $\bar{\partial}_b G_{q,t} : L_{0,q}^2(M) \rightarrow L_{0,q+1}^2(M)$ and $G_{q,t} \bar{\partial}_b : L_{0,q-1}^2(M) \rightarrow L_{0,q}^2(M)$ are continuous (with respect to $\langle |\cdot| \rangle_{t\lambda}$). Additionally,*

- $\bar{\partial}_b G_{q,t} : H_{0,q}^s(M) \rightarrow H_{0,q+1}^s(M)$ and $G_{q,t} \bar{\partial}_b : H_{0,q-1}^s(M) \rightarrow H_{0,q}^s(M)$ are continuous (with respect to $\langle |\Lambda^s \cdot| \rangle_{t\lambda}$) if $t \geq T_s$.
- (viii) The Szegő projections $S_{q,t} = I - \bar{\partial}_b^{*,t} \bar{\partial}_b G_{q,t}$ and $S_{q-1,t} = I - \bar{\partial}_b^{*,t} G_{q,t} \bar{\partial}_b$ are continuous on $L_{0,q}^2(M)$ and $L_{0,q-1}^2(M)$, respectively and with respect to $\langle |\cdot| \rangle_{t\lambda}$. Additionally, if $t \geq T_s$, then $S_{q,t}$ and $S_{q-1,t}$ are continuous on $H_{0,q}^s$ and $H_{0,q-1}^s$ (with respect to $\langle |\Lambda^s \cdot| \rangle_{t\lambda}$), respectively.
- (ix) If $\tilde{q} = q$ or $q+1$ and $\alpha \in H_{0,q}^s(M)$ so that $\bar{\partial}_b \alpha = 0$ and $\alpha \perp \mathcal{H}_t^{\tilde{q}}$ (with respect to $\langle |\cdot| \rangle_{t\lambda}$), then there exists $u \in H_{0,\tilde{q}-1}^s(M)$ so that
- $$\bar{\partial}_b u = \alpha;$$
- (x) If $\tilde{q} = q$ or $q+1$ and $\alpha \in C_{0,\tilde{q}}^\infty(M)$ satisfies $\bar{\partial}_b \alpha = 0$ and $\alpha \perp \mathcal{H}_t^{\tilde{q}}$ (with respect to $\langle |\cdot| \rangle_{t\lambda}$), then there exists $u \in C_{0,\tilde{q}-1}^\infty(M)$ so that
- $$\bar{\partial}_b u = \alpha.$$

Turning to the proof of Theorem 1.6, a consequence of Theorem 4.1 and (4.6) is that $\bar{\partial}_b : L_{0,\tilde{q}}^2(M) \rightarrow L_{0,\tilde{q}+1}^2(M)$ has closed range, $\tilde{q} = q$ or $q-1$. Functional analysis shows that the L^2 -adjoint operators $\bar{\partial}_b^* : L_{0,\tilde{q}+1}^2(M) \rightarrow L_{0,\tilde{q}}^2(M)$ also has closed range [Hör65, Theorem 1.1.1]. Additionally, the finite dimensionality of $\mathcal{H}_t^q(M)$ combined with the closed range of $\bar{\partial}_b$ on $L_{0,\tilde{q}}^2(M)$, $\tilde{q} = q, q-1$ implies the finite dimensionality of the unweighted space of harmonic forms $\mathcal{H}_{0,q}(M)$. While this fact is likely well-known, Straube and Raich give a proof in [RS08, p.772].

The cases $q = 0$ and $q = n$ (when $n \geq 2$) follow easily from the formulas $G_0 = \bar{\partial}_b^* G_1^2 \bar{\partial}_b$ and $G_n = \bar{\partial}_b G_{n-1}^2 \bar{\partial}_b^*$ and the already proven parts of the theorem. This concludes the proof of Theorem 1.6.

5. GLOBAL HYPOELLIPTICITY OF \square_b AND THE PROOF OF THEOREM 1.5

5.1. A weak compactness estimate for \square_b . In this section, we assume: i) for any $\epsilon > 0$ there exist a vector T_ϵ transversal to $T^{1,0}M \oplus T^{0,1}M$ such that $0 < c_1 < \gamma(T_\epsilon) < c_2$ uniformly in ϵ and ii) there exists a covering $\{U_\eta\}$ by local patches such that on each $U := U_\eta$ there exists $\lambda_\epsilon := \lambda_\epsilon^\eta$ so that λ_ϵ is uniformly bounded and

$$\langle (\mathcal{L}_{\lambda_\epsilon} + A_\epsilon d\gamma) \lrcorner u, \bar{u} \rangle \geq \frac{1}{\epsilon} |\alpha_\epsilon|^2 |u|^2$$

holds on U for all $(0, q_0)$ -forms $u \in C_{0,q}^\infty(M)$. Here, $\alpha_\epsilon = -\{\text{Lie}\}_{T_\epsilon}(\gamma)$. In Section 4, we proved estimates for weighted operators in Sobolev spaces. Now, under our stronger assumption of the existence of T_ϵ , we will prove estimates in Sobolev spaces for the unweighted system $(\bar{\partial}_b, \bar{\partial}_b^*)$. In order to do that, we use the composition weight $\phi = \chi(\lambda_\epsilon)$ for a smooth function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ chosen later but satisfying $\dot{\chi}, \ddot{\chi} > 0$. By the definition of the Levi form, it follows that

$$\mathcal{L}_{\chi(\lambda_\epsilon)} = \dot{\chi} \mathcal{L}_{\lambda_\epsilon} + \ddot{\chi} \partial_b \lambda_\epsilon \wedge \bar{\partial}_b \lambda_\epsilon,$$

and hence

$$\begin{aligned} \langle (\mathcal{L}_{\chi(\lambda_\epsilon)} + \dot{\chi} A_\epsilon d\gamma) \lrcorner u, \bar{u} \rangle &= \dot{\chi} \langle (\mathcal{L}_{\lambda_\epsilon} + A_\epsilon d\gamma) \lrcorner u, \bar{u} \rangle + \ddot{\chi} |\partial_b \lambda_\epsilon \lrcorner u|^2 \\ &\geq \frac{1}{\epsilon} |\sqrt{\dot{\chi}} \alpha_\epsilon|^2 |u|^2 + \ddot{\chi} |\partial_b \lambda_\epsilon \lrcorner u|^2. \end{aligned} \quad (5.1)$$

We also notice that

$$|\bar{\partial}_{b, \chi(\phi^\epsilon)}^* u|^2 \leq 2|\bar{\partial}_b^* u|^2 + 2\dot{\chi}^2 |\partial_b \lambda_\epsilon \lrcorner u|^2. \quad (5.2)$$

Now we use Corollary, 3.4(i) for $\phi := \chi(\lambda_\epsilon)$, $A := \dot{\chi}(\lambda_\epsilon) A_\epsilon$ and plug in (5.1) and (5.2) into (3.8) to obtain

$$\begin{aligned} c_\epsilon \|\zeta' \tilde{\Psi}_A^0 \zeta u\|_{L^2}^2 + c \left(\|\tilde{\zeta} \Psi_A^+ \zeta u\|_{L_\phi^2}^2 + 2\|\dot{\chi} \partial_b \lambda_\epsilon \lrcorner \tilde{\zeta} \Psi_A^+ \zeta u\|_{L_\phi^2}^2 + \|\bar{\partial}_b \tilde{\zeta} \Psi_A^+ \zeta u\|_{L_\phi^2}^2 + \|\bar{\partial}_{b, \phi}^* \tilde{\zeta} \Psi_A^+ \zeta u\|_{L_\phi^2}^2 \right) \\ \geq \frac{1}{\epsilon} \|\sqrt{\dot{\chi}} |\alpha_\epsilon| \tilde{\zeta} \Psi_A^+ \zeta u\|_{L_\phi^2}^2 + \|\sqrt{\dot{\chi}} \partial_b \lambda_\epsilon \lrcorner \tilde{\zeta} \Psi_A^+ \zeta u\|_{L_\phi^2}^2 \end{aligned} \quad (5.3)$$

for any $u \in C_{0,q}^\infty(M)$ with $q = q_0, \dots, n$. The function λ_ϵ is uniformly bounded, so we may assume that $|\lambda_\epsilon| \leq 1$. Thus, if we choose $\chi(t) = \frac{1}{2c} e^{t-1}$ then $\ddot{\chi}(t) \geq 2c\dot{\chi}^2(t)$ for $|t| \leq 1$, and hence we can absorb $2c\|\dot{\chi} \partial_b \lambda_\epsilon \lrcorner \tilde{\zeta} \Psi_A^+ \zeta u\|_{L_\phi^2}^2$ by the RHS. By this choice of χ , we also get a uniform bound for $e^{-\chi}$ and $\dot{\chi} \geq \frac{1}{2e^2 c}$. Consequently, we can remove the weight from both sides of (5.3) and obtain

$$\begin{aligned} c_\epsilon \|\zeta' \tilde{\Psi}_A^0 \zeta u\|_{L^2}^2 + c \left(\|\tilde{\zeta} \Psi_A^+ \zeta u\|_{L^2}^2 + \|\bar{\partial}_b \tilde{\zeta} \Psi_A^+ \zeta u\|_{L^2}^2 + \|\bar{\partial}_b^* \tilde{\zeta} \Psi_A^+ \zeta u\|_{L^2}^2 \right) \\ \geq \frac{1}{\epsilon} \|\alpha_\epsilon| \tilde{\zeta} \Psi_A^+ \zeta u\|_{L^2}^2 \end{aligned} \quad (5.4)$$

for any $u \in C_{0,q}^\infty(M)$ with $q \geq q_0$.

To bound the Ψ^- terms, we cannot use an analogous argument. The problem is that there is no $|\partial_b \lambda_\epsilon \lrcorner u|^2$ term to absorb unwanted terms. Indeed,

$$\begin{aligned} &\langle [\text{Tr}(\mathcal{L}_{\chi(\lambda_\epsilon)}) + \dot{\chi} A \text{Tr}(d\gamma)] \times u, \bar{u} \rangle - \langle (\mathcal{L}_{\chi(\lambda_\epsilon)} + \dot{\chi} A_\epsilon d\gamma) \lrcorner u, \bar{u} \rangle \\ &= \dot{\chi} \langle [\text{Tr}(\mathcal{L}_{\lambda_\epsilon}) + A_\epsilon \text{Tr}(d\gamma)] \times u, \bar{u} \rangle - \langle (\mathcal{L}_{\lambda_\epsilon} + A_\epsilon d\gamma) \lrcorner u, \bar{u} \rangle \\ &= \ddot{\chi} \langle \text{Tr}(\partial_b \lambda_\epsilon \wedge \bar{\partial}_b \lambda_\epsilon) \times u, \bar{u} \rangle - |\partial_b \lambda_\epsilon \lrcorner u|^2 \\ &\geq \frac{1}{\epsilon} |\sqrt{\dot{\chi}} \alpha_\epsilon|^2 |u|^2 + \ddot{\chi} |\bar{\partial}_b \lambda_\epsilon \wedge u|^2; \end{aligned} \quad (5.5)$$

and the $|\bar{\partial}_b \lambda_\epsilon \wedge u|$ cannot absorb $|\partial_b \lambda_\epsilon \lrcorner u|$ in general. Instead, we can obtain the estimate for Ψ^- by a Hodge-* argument (see [Koh02, Kha16]). Indeed, using the ideas in [Kha16, Theorem 5], it follows that (5.4) is equivalent to

$$\begin{aligned} c_\epsilon \|\zeta' \tilde{\Psi}_A^0 \zeta u\|_{L^2}^2 + c \left(\|\tilde{\zeta} \Psi_A^- \zeta u\|_{L^2}^2 + \|\bar{\partial}_b \tilde{\zeta} \Psi_A^- \zeta u\|_{L^2}^2 + \|\bar{\partial}_b^* \tilde{\zeta} \Psi_A^- \zeta u\|_{L^2}^2 \right) \\ \geq \frac{1}{\epsilon} \|\alpha_\epsilon| \tilde{\zeta} \Psi_A^- \zeta u\|_{L^2}^2 \end{aligned} \quad (5.6)$$

for any $u \in C_{0,q}^\infty(M)$ with $q \leq n - q_0$.

To obtain our global estimates, we use (5.4) and (5.6) on each local patch U_η of the covering $\{U_\eta\}$, together with the elliptic estimate of the Ψ^0 -terms and

$$\sum_{\eta} \left(\|[\bar{\partial}_b, \tilde{\zeta} \Psi_A^\pm \zeta] u\|_{L^2}^2 + \|[\bar{\partial}_b, \tilde{\zeta} \Psi_A^\pm \zeta] u\|_{L^2}^2 \right) \leq c \left(\|u\|_{L^2}^2 + \|\bar{\partial}_b u\|_{L^2}^2 + \|\bar{\partial}_b^* u\|_{L^2}^2 \right) + c_\epsilon \|u\|_{H^{-1}}^2.$$

We then see

$$\frac{1}{\epsilon} \|\alpha_\epsilon |u|\|_{L^2}^2 \leq c \left(\|\bar{\partial}_b u\|_{L^2}^2 + \|\bar{\partial}_b^* u\|_{L^2}^2 + \|u\|_{L^2}^2 \right) + c_\epsilon \|u\|_{H^{-1}}^2 \quad (5.7)$$

for any $u \in C_{0,q}^\infty(M)$ with $q_0 \leq q \leq n - q_0$. If M admits a strictly CR-plurisubharmonic function on $(0, q_0)$ -forms, we have already proved that

$$\|u\|_{L^2}^2 \leq c \left(\|\bar{\partial}_b u\|_{L^2}^2 + \|\bar{\partial}_b^* u\|_{L^2}^2 \right) + c' \|u\|_{H^{-1}}^2$$

for any $u \in C_{0,q}^\infty(M)$ with $q_0 \leq q \leq n - q_0$. Thus, we have the following theorem.

Theorem 5.1. *Assume that the hypothesis of Theorem 1.1 holds. Then for any $\epsilon > 0$ there exist a vector T_ϵ and a constants $c_\epsilon > 0$ such that $c_1 \leq \gamma(T_\epsilon) \leq c_2$ uniformly in ϵ and*

$$\frac{1}{\epsilon} \|\alpha_\epsilon |u|\|_{L^2}^2 + \|u\|_{L^2}^2 \leq c \left(\|\bar{\partial}_b u\|_{L^2}^2 + \|\bar{\partial}_b^* u\|_{L^2}^2 \right) + c_\epsilon \|u\|_{H^{-1}}^2 \quad (5.8)$$

holds for all $u \in C_{0,q}^\infty(M)$ with $q_0 \leq q \leq n - q_0$ and $\alpha_\epsilon = -\{Lie\}_{T_\epsilon}(\gamma)$.

5.2. Global hypoellipticity for \square_b . Let $s \geq 0$ be an integer and D^s denote a differential operator of order s , $\nabla_b = (\nabla_b^{1,0}, \nabla_b^{0,1})$ where $\nabla_b^{1,0} = (L_1, \dots, L_n)$ and $\nabla_b^{0,1} = (\bar{L}_1, \dots, \bar{L}_n)$. By the Kohn-Morrey-Hörmander inequality, for $s \geq 1$, $\epsilon > 0$ there exists $c_{\epsilon,s} > 0$ such that

$$\begin{aligned} \|\nabla_b u\|_{H^{s-1}}^2 &\leq c_s \left(\|\nabla_b D^{s-1} u\|_{L^2}^2 + \|u\|_{H^{s-1}}^2 \right) \\ &\leq c_s \left(\|\bar{\partial}_b D^{s-1} u\|_{L^2}^2 + \|\bar{\partial}_b^* D^{s-1} u\|_{L^2}^2 + \|u\|_{H^{s-1}}^2 + \|u\|_{H^s} \|u\|_{H^{s-1}} \right) \\ &\leq c_s \left(\|\bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* u\|_{H^{s-1}}^2 \right) + c_{s,\epsilon} \|u\|_{H^{s-1}}^2 + \epsilon \|u\|_{H^s}^2, \end{aligned} \quad (5.9)$$

Let T_ϵ be a global, purely imaginary vector field transversal to $T^{1,0}M \oplus T^{0,1}M$ such that $0 < c_1 \leq \gamma(T_\epsilon) \leq c_2$. There exist a function b_ϵ and a vector field $X_\epsilon \in T^{1,0}M \oplus T^{0,1}M$ such that $0 < c_1 \leq b_\epsilon \leq c_2$ and

$$T = b_\epsilon T_\epsilon + X_\epsilon$$

This implies that for any $s \geq 1$, there exist constants $c > 0$ and $c_{\epsilon,s} > 0$ such that

$$\|T^s u\|_{L^2}^2 \leq c \|T_\epsilon^s u\|_{L^2}^2 + c_{\epsilon,s} \left(\|\nabla_b u\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2 \right) + \epsilon \|u\|_{H^s}^2 \quad (5.10)$$

From (5.9), (5.10) and $\|u\|_{H^s}^2 \leq c_s \|\nabla_b u\|_{L^2}^2 + c \|T^s u\|_{L^2}^2$, we get the reduction from D^s to T_ϵ^s by the inequality

$$\|u\|_{H^s}^2 \leq c_{s,\epsilon} \left(\|\bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2 \right) + c \|T_\epsilon^s u\|_{L^2}^2. \quad (5.11)$$

Moreover, since

$$\|\bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* u\|_{H^{s-1}}^2 \leq c \|\square_b u\|_{H^{s-2}} \|u\|_{H^s} + c_s \|u\|_{H^{s-1}}^2$$

it follows that

$$\|u\|_{H^s}^2 \leq c_{s,\epsilon} (\|\square_b u\|_{H^{s-2}}^2 + \|u\|_{H^{s-1}}^2) + c \|T_\epsilon^s u\|_{L^2}^2. \quad (5.12)$$

Denote by $\alpha_\epsilon^{1,0}$ and $\alpha_\epsilon^{0,1}$ the $(1,0)$ -part and $(0,1)$ -part of the real 1-form $\alpha_\epsilon := -\{\text{Lie}\}_{T_\epsilon}(\gamma)$. Now we express $\alpha_\epsilon^{1,0}$ and $\alpha_\epsilon^{0,1}$ in a local basis. Let $\alpha_{\epsilon,j}$ be the T -component of the commutator $[T_\epsilon, L_j]$. Then

$$\alpha_\epsilon^{1,0}(L_j) = -(\{\text{Lie}\}_{T_\epsilon}(\gamma))(L_j) = -(T_\epsilon \gamma(L_j) - \gamma([T_\epsilon, L_j])) = \gamma([T_\epsilon, L_j]) = \alpha_{\epsilon,j} \quad (5.13)$$

and hence $\alpha_\epsilon^{1,0} = \sum_{j=1}^n \alpha_{\epsilon,j} \omega_j$ and $\alpha_\epsilon^{0,1} = \overline{\alpha_\epsilon^{1,0}} = \sum_{j=1}^n \overline{\alpha_{\epsilon,j}} \bar{\omega}_j$. The commutators $[\bar{\partial}_b, T_\epsilon]$, $[\bar{\partial}_b^*, T_\epsilon]$ and the forms $\alpha_\epsilon^{1,0}$ and $\alpha_\epsilon^{0,1}$ are related by

$$[\bar{\partial}_b, T_\epsilon] = \alpha_\epsilon^{0,1} \wedge T + \tilde{\mathcal{X}}_\epsilon = b_\epsilon \alpha_\epsilon^{0,1} \wedge T_\epsilon + \mathcal{X}_\epsilon;$$

$$[\bar{\partial}_b^*, T_\epsilon] = -\alpha_\epsilon^{1,0} \lrcorner T + \tilde{\mathcal{Y}}_\epsilon = -b_\epsilon \alpha_\epsilon^{1,0} \lrcorner T_\epsilon + \mathcal{Y}_\epsilon.$$

Here, $\mathcal{X}_\epsilon : C_{0,q}^\infty(M) \rightarrow C_{0,q+1}^\infty(M)$ and $\mathcal{Y}_\epsilon : C_{0,q}^\infty(M) \rightarrow C_{0,q-1}^\infty(M)$ so that $\|\mathcal{X}_\epsilon u\|_{L^2} \leq c_\epsilon \|\nabla_b u\|_{L^2}$ and $\|\mathcal{Y}_\epsilon u\|_{L^2} \leq c_\epsilon \|\nabla_b u\|_{L^2}$. In general, for any $s \geq 1$, there exist $\mathcal{X}_{s,\epsilon} : C_q^\infty(M) \rightarrow C_{q+1}^\infty(M)$ and $\mathcal{Y}_{s,\epsilon} : C_q^\infty(M) \rightarrow C_{q-1}^\infty(M)$ such that

$$\begin{aligned} [\bar{\partial}_b, T_\epsilon^s] &= s b_\epsilon \alpha_\epsilon^{0,1} \wedge T_\epsilon^s + \mathcal{X}_{s,\epsilon}; \\ [\bar{\partial}_b^*, T_\epsilon^s] &= -s b_\epsilon \alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s + \mathcal{Y}_{s,\epsilon}, \end{aligned} \quad (5.14)$$

and $\|\mathcal{X}_{s,\epsilon} u\|_{L^2} + \|\mathcal{Y}_{s,\epsilon} u\|_{L^2} \leq c_{s,\epsilon} \|\nabla_b u\|_{H^{s-1}}$. Now we are ready to prove *a priori* estimates and the estimates for elliptic regulation.

Theorem 5.2. *Assume that for any $\epsilon > 0$ there exist a vector T_ϵ and a constants $c, c_1, c_2, c_\epsilon > 0$ such that $c_1 \leq \gamma(T_\epsilon) \leq c_2$ uniformly in ϵ and*

$$\frac{1}{\epsilon} \| \{\text{Lie}\}_{T_\epsilon}(\gamma) u \|_{L^2}^2 + \|u\|_{L^2}^2 \leq c (\|\bar{\partial}_b u\|_{L^2}^2 + \|\bar{\partial}_b^* u\|_{L^2}^2) + c_\epsilon \|u\|_{H^{-1}}^2 \quad (5.15)$$

holds for all $u \in C_{0,q}^\infty(M)$ with $q_0 \leq q \leq n - q_0$. Then

$$\|u\|_{H^s}^2 \leq c_s (\|\bar{\partial}_b u\|_{H^s}^2 + \|\bar{\partial}_b^* u\|_{H^s}^2 + \|u\|_{L^2}^2) \quad (5.16)$$

$$\|\bar{\partial}_b u\|_{H^s}^2 + \|\bar{\partial}_b^* u\|_{H^s}^2 \leq c_s (\|\square_b u\|_{H^s}^2 + \|u\|_{L^2}^2) \quad (5.17)$$

$$\|\bar{\partial}_b \bar{\partial}_b^* u\|_{H^s}^2 + \|\bar{\partial}_b^* \bar{\partial}_b u\|_{H^s}^2 \leq c_s (\|\square_b u\|_{H^s}^2 + \|u\|_{L^2}^2) \quad (5.18)$$

for all $u \in C_{0,q}^\infty(M)$ and nonnegative $s \in \mathbb{Z}$. Furthermore, for any $s \in \mathbb{N}$, there exists $\delta_s > 0$ such that

$$\|u\|_{H^s}^2 \leq c_s (\|\square_b^\delta u\|_{H^s}^2 + \|u\|^2) \quad (5.19)$$

holds for all $u \in C_{0,q}^\infty(M)$ uniformly in $\delta \in (0, \delta_s)$. The operator $\square_b^\delta = \square_b + \delta (T^ T + \text{Id})$.*

Proof. We prove (5.16) by inducting in s . It is easy to see that (5.16) holds for $s = 0$. We now assume that (5.16) holds for $s - 1$ with $s \geq 1$ and we are going to prove this estimate still holds on level s . We first fix $\epsilon > 0$ independent of s . We start with (5.15) with u replaced by $T_\epsilon^s u$ and use the equality

$$|\alpha_\epsilon|^2 |u|^2 = 2 \left(|\alpha_\epsilon^{0,1} \wedge u|^2 + |\alpha_\epsilon^{1,0} \lrcorner u|^2 \right)$$

to see

$$\begin{aligned} \|T_\epsilon^s u\|_{L^2}^2 + \frac{1}{\epsilon} \left(\|\alpha_\epsilon^{0,1} \wedge T_\epsilon^s u\|_{L^2}^2 + \|\alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s u\|_{L^2}^2 \right) \\ \leq c \left(\|\bar{\partial}_b T_\epsilon^s u\|_{L^2}^2 + \|\bar{\partial}_b^* T_\epsilon^s u\|_{L^2}^2 \right) + c_\epsilon \|T_\epsilon^s u\|_{H^{-1}}^2. \end{aligned} \quad (5.20)$$

From (5.14), we have

$$\begin{aligned} \|\bar{\partial}_b T_\epsilon^s u\|_{L^2}^2 &\leq 3 \|T_\epsilon^s \bar{\partial}_b u\|_{L^2}^2 + cs^2 \|\alpha_\epsilon^{0,1} \wedge T_\epsilon^s u\|_{L^2}^2 + c_{\epsilon,s} \|\nabla_b u\|_{H^{s-1}}^2, \\ \|\bar{\partial}_b^* T_\epsilon^s u\|_{L^2}^2 &\leq 3 \|T_\epsilon^s \bar{\partial}_b^* u\|_{L^2}^2 + cs^2 \|\alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s u\|_{L^2}^2 + c_{\epsilon,s} \|\nabla_b u\|_{H^{s-1}}^2, \end{aligned} \quad (5.21)$$

where we have used that b_ϵ is uniformly bounded. However, for each s , there exists ϵ_s such that for any $\epsilon < \epsilon_s$ the expression $s^2 \left(\|\alpha_\epsilon^{0,1} \wedge T_\epsilon^s u\|_{L^2}^2 + \|\alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s u\|_{L^2}^2 \right)$ can be absorbed. Thus

$$\begin{aligned} \|T_\epsilon^s u\|_{L^2}^2 + \frac{1}{\epsilon} \left(\|\alpha_\epsilon^{0,1} \wedge T_\epsilon^s u\|_{L^2}^2 + \|\alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s u\|_{L^2}^2 \right) \\ \leq c \left(\|T_\epsilon^s \bar{\partial}_b u\|_{L^2}^2 + \|T_\epsilon^s \bar{\partial}_b^* u\|_{L^2}^2 \right) + c_{s,\epsilon} \left(\|\nabla_b u\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2 \right). \end{aligned} \quad (5.22)$$

Combining with (5.9) and (5.11), it follows

$$\|u\|_{H^s}^2 \leq c \left(\|T_\epsilon^s \bar{\partial}_b u\|_{L^2}^2 + \|T_\epsilon^s \bar{\partial}_b^* u\|_{L^2}^2 \right) + c_{s,\epsilon} \left(\|\bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2 \right) \quad (5.23)$$

Thus the first *a priori* estimate follows by using the inductive hypothesis for $\|u\|_{H^{s-1}}^2$.

For the second *a priori* estimate, it follows from (5.14) that

$$\begin{aligned} \|T_\epsilon^s \bar{\partial}_b u\|_{L^2}^2 &= (T_\epsilon^s \bar{\partial}_b^* \bar{\partial}_b u, T_\epsilon^s u) + ([\bar{\partial}_b^*, T_\epsilon^s] \bar{\partial}_b u, T_\epsilon^s u) + (T_\epsilon^s \bar{\partial}_b u, [T_\epsilon^s, \bar{\partial}_b] u) \\ &= (T_\epsilon^s \bar{\partial}_b^* \bar{\partial}_b u, T_\epsilon^s u) - s(b_\epsilon \alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s \bar{\partial}_b u, T_\epsilon^s u) + (\mathcal{Y}_{s,\epsilon} \bar{\partial}_b u, T_\epsilon^s u) \\ &\quad + s(T_\epsilon^s \bar{\partial}_b u, b_\epsilon \alpha_\epsilon^{0,1} \wedge T_\epsilon^s u) + (T_\epsilon^s \bar{\partial}_b u, \mathcal{X}_{s,\epsilon} u); \end{aligned} \quad (5.24)$$

and similarly,

$$\begin{aligned} \|T_\epsilon^s \bar{\partial}_b^* u\|_{L^2}^2 &= (T_\epsilon^s \bar{\partial}_b \bar{\partial}_b^* u, T_\epsilon^s u) + s(b_\epsilon \alpha_\epsilon^{0,1} \wedge T_\epsilon^s \bar{\partial}_b^* u, T_\epsilon^s u) + (\mathcal{X}_{s,\epsilon} \bar{\partial}_b^* u, T_\epsilon^s u) \\ &\quad - s(T_\epsilon^s \bar{\partial}_b^* u, b_\epsilon \alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s u) + (T_\epsilon^s \bar{\partial}_b^* u, \mathcal{Y}_{s,\epsilon} u), \end{aligned} \quad (5.25)$$

Next, we sum and use the equality $(\alpha_\epsilon^{0,1} \wedge u, v) = (u, \alpha_\epsilon^{1,0} \lrcorner v)$, the $(sc) - (lc)$ inequality, and the uniform boundedness of b_ϵ to obtain

$$\begin{aligned} & \|T_\epsilon^s \bar{\partial}_b u\|_{L^2}^2 + \|T_\epsilon^s \bar{\partial}_b^* u\|_{L^2}^2 \leq (T_\epsilon^s \square_b u, T_\epsilon^s u) \\ & + c_{s,\epsilon} (\|\nabla_b \bar{\partial}_b u\|_{H^{s-1}}^2 + \|\nabla_b \bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|\nabla_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2) \\ & + sc (\|T_\epsilon^s \bar{\partial}_b u\|_{L^2}^2 + \|T_\epsilon^s \bar{\partial}_b^* u\|_{L^2}^2 + \|T_\epsilon^s u\|_{L^2}^2) + lc s^2 (\|\alpha_\epsilon^{0,1} \wedge T_\epsilon^s u\|_{L^2}^2 + \|\alpha_\epsilon^{1,0} \lrcorner T_\epsilon^s u\|_{L^2}^2) \end{aligned} \quad (5.26)$$

By (5.22), we may absorb the term the last line by choosing $\epsilon < \epsilon_s$ sufficiently small. We can bound $\|u\|_s^2$ with (5.23) (5.26) and observe

$$\|u\|_{H^s}^2 + \|\bar{\partial}_b u\|_{H^s}^2 + \|\bar{\partial}_b^* u\|_{H^s}^2 \leq c(T_\epsilon^s \square_b u, T_\epsilon^s u) + I \quad (5.27)$$

where

$$\begin{aligned} I &= c_{s,\epsilon} (\|\nabla_b \bar{\partial}_b u\|_{H^{s-1}}^2 + \|\nabla_b \bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|\nabla_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2) \\ &\leq c_{s,\epsilon} \left(\|\bar{\partial}_b u\|_s \|\bar{\partial}_b u\|_{H^{s-1}} + \|\bar{\partial}_b^* u\|_s \|\bar{\partial}_b^* u\|_{H^{s-1}} + \|u\|_{H^s} \|u\|_{H^{s-1}} \right. \\ &\quad \left. + \|\bar{\partial}_b \bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* \bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b u\|_{H^{s-1}}^2 + \|\bar{\partial}_b^* u\|_{H^{s-1}}^2 + \|u\|_{H^{s-1}}^2 \right) \\ &\leq sc (\|u\|_{H^s}^2 + \|\bar{\partial}_b u\|_{H^s}^2 + \|\bar{\partial}_b^* u\|_{H^s}^2) + c_{\epsilon,s} lc (\|\square_b u\|_{H^{s-1}}^2 + \|u\|_{L^2}^2) \end{aligned} \quad (5.28)$$

here the second inequality follows by the middle line of (5.9) and the last inequality follows by the inductive hypothesis. Thus,

$$\|u\|_{H^s}^2 + \|\bar{\partial}_b u\|_{H^s}^2 + \|\bar{\partial}_b^* u\|_{H^s}^2 \leq c(T_\epsilon^s \square_b u, T_\epsilon^s u) + c_{\epsilon,s} (\|\square_b u\|_{H^{s-1}}^2 + \|u\|_{L^2}^2) \quad (5.29)$$

The second *a priori* estimate now follows immediately by (5.29). The last *a priori* estimate follows by the second *a priori* estimate and the inequality

$$\begin{aligned} & \|\bar{\partial}_b \bar{\partial}_b^* u\|_{H^s}^2 + \|\bar{\partial}_b \bar{\partial}_b^* u\|_{H^s}^2 = \|\square_b u\|_{H^s}^2 - 2\text{Re}(\Lambda^s \bar{\partial}_b \bar{\partial}_b^* u, \Lambda^s \bar{\partial}_b^* \bar{\partial}_b u) \\ &= \|\square_b u\|_{H^s}^2 - 2\text{Re} \left((\Lambda^s \bar{\partial}_b \bar{\partial}_b \bar{\partial}_b^* u, \Lambda^s \bar{\partial}_b u) + ([\bar{\partial}_b, \Lambda^s] \bar{\partial}_b \bar{\partial}_b^* u, \Lambda^s \bar{\partial}_b u) + (\Lambda^s \bar{\partial}_b \bar{\partial}_b^* u, [\Lambda^s, \bar{\partial}_b^*] \bar{\partial}_b u) \right) \\ &\leq \|\square_b u\|_{H^s}^2 + sc \|\bar{\partial}_b \bar{\partial}_b^* u\|_{H^s}^2 + lc \|\bar{\partial}_b u\|_{H^s}^2. \end{aligned}$$

We now prove (5.19), the estimate that allows us to use the method of elliptic regularization. We first show

$$\|u\|_{H^{s+1}}^2 \leq c(T_\epsilon^s T^* T u, T_\epsilon^s u) + c_{\epsilon,s} (\|\square_b u\|_{H^{s-1}}^2 + \|u\|_{H^s}^2). \quad (5.30)$$

This estimate follows quickly by combining (5.12),

$$\|u\|_{H^{s+1}}^2 \leq c \|T T_\epsilon^s u\|_{L^2}^2 + c_{\epsilon,s} (\|\square_b u\|_{H^{s-1}}^2 + \|u\|_{H^s}^2),$$

and

$$\begin{aligned} \|T T_\epsilon^s u\|^2 &= (T_\epsilon^s T^* T u, T_\epsilon^s u) + ([T^* T, T_\epsilon^s] u, T_\epsilon^s u) \\ &\leq (T_\epsilon^s T^* T u, T_\epsilon^s u) + sc \|u\|_{H^{s+1}}^2 + c_{\epsilon,s} lc \|u\|_{H^s}^2. \end{aligned} \quad (5.31)$$

But (5.30) and (5.29), we get

$$\begin{aligned}
& \|u\|_{H^s}^2 + \|\bar{\partial}_b u\|_{H^s}^2 + \|\bar{\partial}_b^* u\|_{H^s}^2 + \delta \|u\|_{H^{s+1}}^2 \\
& \leq c(T_\epsilon^s((\square_b + \delta(T_\epsilon^* T_\epsilon + I))u, T_\epsilon^s u) + c_{\epsilon,s}(\|\square_b u\|_{H^{s-1}}^2 + \delta \|u\|_{H^s}^2 + \|u\|_{L^2}^2) \\
& \leq c_{\epsilon,s}(\|\square_b^\delta u\|_{H^s}^2 + \|\square_b^\delta u\|_{H^{s-1}}^2 + \|u\|_{L^2}^2) \\
& \quad + \delta c_{\epsilon,s} \|u\|_{H^s}^2 + \delta^2 c_{\epsilon,s} \|u\|_{H^{s+1}}^2,
\end{aligned} \tag{5.32}$$

where we have used that $\|\square_b u\|_{H^{s-1}}^2 \leq 2\|\square_b^\delta u\|_{H^{s-1}}^2 + \delta^2 \|u\|_{H^{s+1}}^2$. Now we fix ϵ depending on s so the above estimates hold, and choose δ_s such that $\delta c_{\epsilon,s}$ and $\delta^2 c_{\epsilon,s}$ are small for any $\delta \leq \delta_s$. Thus, we may absorb the last line and obtain an inequality stronger than the desired inequality. \square

5.3. Proof of Theorem 1.1. (i) *Proof of the global regularity of G_q .* For a given $\varphi \in C_{0,q}^\infty(M)$, we first prove that $G_q \varphi \in C_{0,q}^\infty(M)$ by elliptic regularization using the elliptic perturbation $\square_b^\delta := \square_b + \delta(T^*T + \text{Id})$. First, though, we make one short remark. Since $\mathcal{H}_{0,q}(M)$ is finite dimensional, and all norms on finite dimensional vector spaces are equivalent, it follows that $\|u\|_{L^2} \approx \|u\|_{H^{-1}(M)}$ for any $u \in \mathcal{H}_{0,q}(M)$. From this equivalence and the density of smooth forms, we may conclude that harmonic forms are smooth and $\|u\|_{H^s} \approx \|u\|_{L^2}$ where the equivalence depends on s but is independent of the harmonic $(0, q)$ -form u .

Let $Q_b^\delta(\cdot, \cdot)$ be the quadratic form on $H_{0,q}^1(M)$ defined by

$$Q_b^\delta(u, v) = Q_b(u, v) + \delta((Tu, Tv) + (u, v)) = (\square_b^\delta u, v)$$

By (5.11), we have $\|u\|_1^2 \leq c_\delta Q_b^\delta(u, u)$ for any $u \in H_{0,q}^1(M)$. Consequently, \square_b^δ is a self-adjoint, elliptic operator with inverse G_q^δ . By elliptic theory, we know that if $\varphi \in C_{0,q}^\infty(M)$, then $G_q^\delta \varphi \in C_{0,q}^\infty(M)$. We can therefore use (5.19) with $u = G_q^\delta \varphi$ and estimate

$$\|G_q^\delta \varphi\|_{H^s}^2 \leq c_s(\|\square_b^\delta G_q^\delta \varphi\|_{H^s}^2 + \|G_q^\delta \varphi\|_{L^2}^2) = c_s(\|\varphi\|_{H^s}^2 + \|G_q^\delta \varphi\|_{L^2}^2)$$

where the equality follows from the identity $\square_b^\delta G_q^\delta = \text{Id}$ (since $\text{Ker}(\square_b^\delta) = \{0\}$). By Lemma 5.3, $\|G_q^\delta \varphi\|_{L^2} \leq c\|\varphi\|_{L^2}$ uniformly in δ when $1 \leq q_0 \leq q \leq n - q_0$. Thus, $\|G_q^\delta \varphi\|_{H^s}$ is uniformly bounded and hence there exists a subsequence δ_k and $\tilde{u} \in H_{0,q}^s(M)$ such that $G_q^{\delta_k} \varphi \rightarrow \tilde{u}$ weakly in $H_{0,q}^s(M)$. Consequently, $G_q^{\delta_k} \varphi \rightarrow \tilde{u}$ weakly in the Q_b -norm, which means that if $v \in H_{0,q}^2(M)$, then

$$\lim_{\delta_k \rightarrow 0} Q_b(G_q^{\delta_k} \varphi, v) = Q_b(\tilde{u}, v).$$

On the other hand,

$$Q_b(G_q \varphi, v) = (\varphi, v) = Q_b^\delta(G_q^\delta \varphi, v) = Q_b(G_q^\delta \varphi, v) + \delta((TG_q^\delta \varphi, Tv) + (G_q^\delta \varphi, v))$$

for all $v \in H_{0,q}^2(M)$. It follows that

$$|Q_b((G_q^\delta \varphi - G_q \varphi), v)| \leq \delta \|G_q^\delta \varphi\|_{L^2} \|v\|_2 \leq c\delta \|\varphi\|_{L^2} \|v\|_2 \rightarrow 0 \quad \text{as } \delta \rightarrow 0$$

where we have again used the inequality $\|G_q^\delta \varphi\|_{L^2} \leq c\|\varphi\|_{L^2}$ uniformly in δ . We therefore have $G_q \varphi = \tilde{u} \in H_{0,q}^s(M)$. This holds for arbitrary $s \in \mathbb{N}$, so the Sobolev Lemma implies that $G_q \varphi \in C_{0,q}^\infty(M)$.

(ii) *Proof of the exact regularity of G_q , $\bar{\partial}_b G_q$, $\bar{\partial}_b^* G_q$, $I - \bar{\partial}_b^* \bar{\partial}_b G_q$ and $I - \bar{\partial}_b \bar{\partial}_b^* G_q$.* For $\varphi \in C_{0,q}^\infty(M)$, we use the estimates in Theorem 5.14 with $u = G_q \varphi \in C_{0,q}^\infty(M)$ and observe

$$\begin{aligned} & \|G_q \varphi\|_{H^s}^2 + \|\bar{\partial}_b G_q \varphi\|_{H^s}^2 + \|\bar{\partial}_b^* G_q \varphi\|_{H^s}^2 + \|\bar{\partial}_b^* \bar{\partial}_b G_q \varphi\|_{H^s}^2 + \|\bar{\partial}_b \bar{\partial}_b^* G_q \varphi\|_{H^s}^2 \\ & \leq c_s (\|\square_b G_q \varphi\|_{H^s}^2 + \|G_q \varphi\|_{L^2}^2) = c_s (\|(I - H_q) \varphi\|_{H^s}^2 + \|G_q \varphi\|_{L^2}^2) \\ & \leq c_s (\|\varphi\|_{H^s}^2 + \|H_q \varphi\|_{H^s}^2 + \|G_q \varphi\|_{L^2}^2) \\ & \leq c_s (\|\varphi\|_{H^s}^2 + \|\varphi\|_{L^2}^2) \leq c_s \|\varphi\|_{H^s}^2. \end{aligned} \tag{5.33}$$

We have shown the Sobolev estimate holds for $\varphi \in C_{0,q}^\infty(M)$, and this space is dense in $H_{0,q}^s(M)$. Consequently, since G_q is continuous on $L_{0,q}^2(M)$, so that the Sobolev estimate carries over to $\varphi \in H_{0,q}^s(M)$. This means G_q , $\bar{\partial}_b G_q$, $\bar{\partial}_b^* G_q$, $I - \bar{\partial}_b^* \bar{\partial}_b G_q$, and $I - \bar{\partial}_b \bar{\partial}_b^* G_q$ are exactly regular.

For $G_q \bar{\partial}_b$ and $I - \bar{\partial}_b^* G_q \bar{\partial}_b$, let $\varphi \in C_{0,q-1}^\infty(M)$. By using the estimate (5.29) with $u = G_q \bar{\partial}_b \varphi \in C_{0,q}^\infty(M)$ we obtain

$$\begin{aligned} & \|G_q \bar{\partial}_b \varphi\|_{H^s}^2 + \|\bar{\partial}_b^* G_q \bar{\partial}_b \varphi\|_{H^s}^2 \leq c(T_\epsilon^s \square_b G_q \bar{\partial}_b \varphi, T_\epsilon^s G_q \bar{\partial}_b \varphi) + c_{\epsilon,s} (\|\square_b G_q \bar{\partial}_b \varphi\|_{H^{s-1}}^2 + \|G_q \bar{\partial}_b \varphi\|_{L^2}^2) \\ & \leq c(T_\epsilon^s (I - H_q) \bar{\partial}_b \varphi, T_\epsilon^s G_q \bar{\partial}_b \varphi) + c_{\epsilon,s} (\|(I - H_q) \bar{\partial}_b \varphi\|_{H^{s-1}}^2 + \|\varphi\|_{L^2}^2) \\ & \leq c \left((T_\epsilon^s \varphi, T_\epsilon^s \bar{\partial}_b^* G_q \bar{\partial}_b \varphi) + ([T_\epsilon^s, \bar{\partial}_b] \varphi, T_\epsilon^s G_q \bar{\partial}_b \varphi) + (T_\epsilon^s \varphi, [\bar{\partial}_b^*, T_\epsilon^s] G_q \bar{\partial}_b \varphi) \right. \\ & \quad \left. + (T_\epsilon^* T_\epsilon^s H_q \varphi, T_\epsilon^{s-1} G_q \bar{\partial}_b \varphi) \right) + c_{\epsilon,s} (\|\bar{\partial}_b \varphi\|_{H^{s-1}}^2 + \|\varphi\|_{L^2}^2) \\ & \leq \text{sc} (\|G_q \bar{\partial}_b \varphi\|_{H^s}^2 + \|\bar{\partial}_b^* G_q \bar{\partial}_b \varphi\|_{H^s}^2) + c_\epsilon \text{lc} (\|\varphi\|_{H^s}^2 + \|H_q \varphi\|_{H^{s+1}}^2) \end{aligned} \tag{5.34}$$

Absorbing the sc term by the LHS and using the fact that $\|H_q \bar{\partial}_b \varphi\|_{H^{s+1}} \leq c\|\bar{\partial}_b \varphi\|_{L^2} \leq c\|\varphi\|_1$, we conclude that

$$\|G_q \bar{\partial}_b \varphi\|_{H^s}^2 + \|(I - \bar{\partial}_b^* G_q \bar{\partial}_b) \varphi\|_{H^s}^2 \leq c_s \|\varphi\|_{H^s}^2$$

for all $\varphi \in C_{0,q-1}^\infty(M)$. As in the above argument, this Sobolev estimate also holds for $\varphi \in H_{0,q-1}^s(M)$. We may then prove the exact regularity of $G_q \bar{\partial}_b^*$ and $(I - \bar{\partial}_b G_q \bar{\partial}_b^*)$ for forms of degree $(0, q+1)$ similarly.

Finally, exact regularity of $\bar{\partial}_b^* G_q$, $G_q \bar{\partial}_b$, $\bar{\partial}_b G_q$, $G_q \bar{\partial}_b^*$ implies that $\bar{\partial}_b^* G_q^2 \bar{\partial}_b$ and $\bar{\partial}_b G_q^2 \bar{\partial}_b^*$ are also exactly regular. It is known that on the top degrees the Green operators G_0 and G_n are given by $\bar{\partial}_b^* G_1^2 \bar{\partial}_b$ and $\bar{\partial}_b G_{n-1}^2 \bar{\partial}_b^*$, respectively. Moreover, $\bar{\partial}_b G_0 = G_1 \bar{\partial}_b$, $\bar{\partial}_b^* G_n = G_{n-1} \bar{\partial}_b^*$. Therefore, if $q = 1$ then G_0 , $\bar{\partial}_b G_0$, G_n , $\bar{\partial}_b^* G_n$ are exactly regular.

The proof of Theorem 1.1 is complete, pending the following technical lemma.

Lemma 5.3. *Fix $1 \leq q \leq n-1$. Let M^{2n+1} be an abstract CR manifold that the L^2 basic estimate*

$$\|u\|_{L^2}^2 \leq c(\|\bar{\partial}_b u\|_{L^2}^2 + \|\bar{\partial}_b^* u\|_{L^2}^2) + C\|u\|_{H^{-1}}^2 \quad (5.35)$$

holds for all $u \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}_b^)$. Then $\|G_q^\delta \varphi\| \leq c\|\varphi\|$ uniformly in δ for $\varphi \in L_{0,q}^2(M)$.*

Proof. Fix $\delta > 0$. It suffices to show that

$$\|u\|_{L^2}^2 \leq cQ_b^\delta(u, u) \quad (5.36)$$

holds for some constant $c > 0$ that is independent of $\delta > 0$ and $u \in H_{0,q}^1(M)$. The basic estimate certainly implies that

$$\|u\|_{L^2}^2 \leq cQ_b^\delta(u, u) + c'\|u\|_{H^{-1}}^2,$$

uniformly in δ for all $u \in H_{0,q}^1(M)$. Assume that (5.36) fails. Then there exists u_k with $\|u_k\|_{L^2}^2 = 1$ so that

$$\|u_k\|_{L^2}^2 \geq kQ_b^\delta(u_k, u_k). \quad (5.37)$$

For k sufficiently large, we can use (5.37) and absorb $Q_b^\delta(u_k, u_k)$ by $\|u_k\|^2$ and to prove

$$\|u_k\|_{L^2}^2 \leq 2c'\|u_k\|_{H^{-1}}^2. \quad (5.38)$$

Since $L_{0,q}^2(M)$ is compact in $H_{0,q}^{-1}(M)$, there exists a subsequence u_{k_j} that converges in $H_{0,q}^{-1}(M)$. Thus, (5.38) forces u_{k_j} to converge in $L_{0,q}^2(M)$; and (5.37) forces u_{k_j} to converge in the $Q_b^\delta(\cdot, \cdot)$ -norm as well. The limit u satisfies $\|u\|_{L^2} = 1$. However, a consequence of (5.37) is that $\|u\|_{L^2} = 0$ since $Q_b^\delta(u, u) \geq \delta\|u\|_{L^2}^2$. This is a contradiction and (5.36) holds. \square

5.4. Proof of Theorem 1.5.

Proof of Theorem 1.5. Assume that there exists a global contact form $\tilde{\gamma}$ and a smooth function \tilde{c}_{00} such that the extended Levi matrix $\tilde{\mathcal{M}} := \{\tilde{c}_{ij}\}_{i,j=0}^n$ is positive semidefinite. Thus we can use the Schwarz inequality for the two vectors $u = (0, u_1, \dots, u_n)$, $v = (1, 0, \dots, 0)$ in \mathbb{C}^{n+1} and get

$$\left| \sum_{j=1}^n \tilde{c}_{0j} u_j \right|^2 = \left| \tilde{\mathcal{M}}(u, v) \right|^2 \leq \tilde{\mathcal{M}}(u, u) \tilde{\mathcal{M}}(v, v) = \left| \sum_{i,j=1}^n \tilde{c}_{ij} u_i \bar{u}_j \right| |\tilde{c}_{00}|. \quad (5.39)$$

On the other hand, there exists a smooth function h in M such that $\tilde{\gamma} = e^{-h}\gamma$. Thus,

$$\begin{aligned}
d\tilde{\gamma} &= e^{-h} d\gamma - e^{-h} dh \wedge \gamma \\
&= -e^{-h} \left(\sum_{i,j=1}^n c_{ij} \omega_i \wedge \bar{\omega}_j + \sum_{j=1}^n c_{0j} \gamma \wedge \omega_j + \sum_{j=1}^n \bar{c}_{0j} \gamma \wedge \bar{\omega}_j \right) + e^{-h} \sum_{j=1}^n (L_j(h) \gamma \wedge \omega_j + \bar{L}_j(h) \gamma \wedge \bar{\omega}_j) \\
&= - \sum_{i,j=1}^n e^{-h} c_{ij} \omega_i \wedge \bar{\omega}_j - \sum_{j=1}^n \left(e^{-h} (c_{0j} - L_j(h)) \gamma \wedge \omega_j + e^{-h} (\bar{c}_{0j} - \bar{L}_j(h)) \gamma \wedge \bar{\omega}_j \right) \\
&= - \left(\sum_{i,j=1}^n \tilde{c}_{ij} \omega_i \wedge \bar{\omega}_j + \sum_{j=1}^n \tilde{c}_{0j} \tilde{\gamma} \wedge \omega_j + \sum_{j=1}^n \bar{\tilde{c}}_{0j} \tilde{\gamma} \wedge \bar{\omega}_j \right),
\end{aligned} \tag{5.40}$$

where $\tilde{c}_{ij} = e^{-h} c_{ij}$ for $i, j = 1, \dots, n$ and $\tilde{c}_{0j} = c_{0j} - L_j(h)$. Substituting these \tilde{c}_{ij} into (5.39), we get

$$\left| \sum_{j=1}^n (c_{0j} - L_j(h)) u_j \right|^2 \leq |\tilde{c}_{00}| \left| \sum_{i,j=1}^n \tilde{c}_{ij} u_i \bar{u}_j \right| = |\tilde{c}_{00}| e^{-h} \sum_{i,j=1}^n c_{ij} u_i \bar{u}_j \tag{5.41}$$

holds at any $x \in M$ and any vector $u = (u_1, \dots, u_n) \in \mathbb{C}^n$. Recall that

$$\alpha = -\{\text{Lie}\}_T(\gamma) = - \sum_{j=1}^n (d\gamma(T, L_j) \omega_j + d\gamma(T, \bar{L}_j) \bar{\omega}_j) = \sum_{j=1}^n (c_{0j} \omega_j + \bar{c}_{0j} \bar{\omega}_j),$$

(where the last inequality follows by (2.1)). If $L = \sum_j u_j L_j \in T^{1,0}M$ then we rewrite (5.41) as

$$|\alpha(L) - dh(L)| \leq cd\gamma(L \wedge \bar{L}).$$

This calculation implies that α is exact on the null space of Levi form. The argument from [SZ15, Proposition 1] then finishes the proof. Straube and Zeytuncu's work shows that if α is exact on the null space of the Levi form then for any ϵ , then we can find a vector T_ϵ traversal to $T^{0,1}M \oplus T^{0,1}M$ such that the T -component of $[T_\epsilon, L]$ is less than ϵ for any unit vector field $L \in T^{1,0}$. Their result is stated for embedding manifolds in \mathbb{C}^N but a careful examination of the proof reveals that embeddedness is an unnecessary assumption with their proof. \square

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